Mixing angles of quarks and leptons in Quantum Field Theory

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MIXING ANGLES OF QUARKS AND LEPTONS IN QUANTUM FIELD THEORY

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Abstract: Arguments coming from Quantum Field Theory are supplemented with a 1-loop perturbative calculation to settle the non-unitarity of mixing matrices linking renormalized mass eigenstates to bare flavor states for non-degenerate coupled fermions. We simultaneously diagonalize the kinetic and mass terms and counterterms in the renormalized Lagrangian. $SU(2)_L$ gauge invariance constrains the mixing matrix in charged currents of renormalized mass states, for example the Cabibbo matrix, to stay unitary. Leaving aside CP violation, we observe that the mixing angles exhibit, within experimental uncertainty, a very simple breaking pattern of $SU(2)_f$ horizontal symmetry linked to the algebra of weak neutral currents, the origin of which presumably lies beyond the Standard Model. It concerns: on one hand, the three quark mixing angles; on the other hand, a neutrino-like pattern in which θ_{23} is maximal and $\tan(2\theta_{12}) = 2$. The Cabibbo angle fulfills the condition $\tan(2\theta_c) = 1/2$ and θ_{12} for neutrinos satisfies accordingly the "quark-lepton complementarity condition" $\theta_c + \theta_{12} = \pi/4$. $\theta_{13} = \pm 5.7\, 10^{-3}$ are the only values obtained for the third neutrino mixing angle that lie within present experimental bounds. Flavor symmetries, their breaking by a non-degenerate mass spectrum, and their entanglement with the gauge symmetry, are scrutinized; the special role of flavor rotations as a very mildly broken symmetry of the Standard Model is outlined.

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1 Introduction

In the Standard Model of electroweak interactions [1], universality (we think in particular of gauge neutral currents) is very well verified for mass states, which are the observed and propagating states; non-diagonal transitions (for example $d \leftrightarrow s$ transitions – see Fig.1 –) as well as non-diagonal neutral currents and small violations of universality are generated at 1-loop by charged weak currents and the Cabibbo mixing. This empirical fact is consistent with the gauge Lagrangian for neutral currents being controlled, in mass space, by the unit matrix (this will be justified later on more precise grounds). This work, motivated by results of [2] and [3], which are summarized below, rests on the fact that, in Quantum Field Theory (QFT) of non-degenerate coupled systems like fermions, the unit matrix controlling neutral currents in mass space does not translate *a priori* unchanged when one goes from mass states to flavor states. We show that neutral gauge currents exhibit, in bare flavor space, peculiar and regular structures related to flavor transformations and symmetries.

We have shown in [2] that, in QFT, mixing matrices linking bare flavor to renormalized mass eigenstates for non-degenerate coupled systems should never be parametrized as unitary. Indeed, assuming that the renormalized (q^2 dependent, effective) quadratic Lagrangian is hermitian at any q^2, different mass eigenstates, which correspond to different values of q^2 (poles of the renormalized propagator), belong in general to different orthonormal bases [1] [2]; this is the main property pervading the present work. We recover this result in section 2 from perturbative arguments, through the introduction of counterterms (that we shall call hereafter Shabalin's counterterms) canceling, at 1-loop, on mass-shell $d \leftrightarrow s$ transitions and equivalent [4].

Assuming, for mass states, universality of diagonal neutral currents and absence of their non-diagonal counterparts, these two properties can only be achieved for bare flavor states in two cases [3]: "Cabibbo-like" mixing angles (the standard case), and a set of discrete solutions, unnoticed in the customary approach, including in particular the so-called maximal mixing $\pi/4 \pm k\pi/2$. While, for any of these, one recovers a unitary mixing matrix, the very small departure from unitarity expected because of mass splittings manifests itself as tiny violations of the two previous conditions in the bare mass basis: universality gets slightly violated and flavor changing neutral currents (FCNC's) arise. We empirically found [3] that these violations obey a very precise pattern: in the neighborhood of a Cabibbo-like solution, they become of equal strength for a mixing angle extremely close to its measured value

$$\tan(2\theta_c) = \frac{1}{2}. \tag{1}$$

This success was a encouragement to go further in this direction. We present below the outcome of our investigation in the case of three generations of fermions. The resulting intricate system of trigonometric equations has been analytically solved by successive approximations, starting from configurations in which θ_{13} is vanishing. We will see that this approximation, obviously inspired by the patterns of mixing angles determined from experimental measurements, turns out to be a very good one. Indeed, we show, without exhibiting all the solutions of our equations, that the presently observed patterns of quarks as well as of neutrinos, do fulfill our criterion with a precision smaller than experimental uncertainty.. While the three angles of the Cabibbo-Kobayashi-Maskawa (CKM) solution are "Cabibbo-like", the neutrino-like solution

$$\begin{aligned} \tan(2\theta_{12}) &= 2 \Leftrightarrow \theta_{12} \approx 31.7^o, \\ \theta_{23} &= \frac{\pi}{4}, \\ \theta_{13} &= \pm 5.7\,10^{-3} \text{ or } \theta_{13} = \pm 0.2717 \end{aligned} \tag{2}$$

[1] Since, at any *given* q^2, the set of eigenstates of the renormalized quadratic Lagrangian form an orthonormal basis, the mixing matrix with all its elements evaluated at this q^2 is unitary and the unitarity of the theory is never jeopardized.

[2] Special cases can occur, in which two coupled states with different masses can be orthogonal: this would be the case of neutral kaons in a world where they are stable and where CP symmetry is not violated; the mass eigenstates are then the orthogonal K_1^0 and K_2^0 mesons [5].

[3] For two generations, one is led to introduce two mixing angles to parametrize each 2×2 non-unitary mixing matrix.

is of a mixed type, where θ_{23} is maximal while θ_{12} and θ_{13} are Cabibbo-like.

Two significant features in these results must be stressed. First, the values for the third neutrino mixing angle θ_{13} given in (2) are the only ones which lie within present (loose) experimental bounds; only two solutions satisfy this constraint: a very small value $\theta_{13} \sim V_{ub} \sim$ a few 10^{-3}, and a rather "large" one, at the opposite side of the allowed range (it actually lies slightly beyond present experimental upper limit). Secondly, our procedure yields in an exact, though quite simple way, the well-known "quark-lepton complementarity relation" [6] for 1-2 mixing:

$$\theta_{12} + \theta_c = \frac{\pi}{4}, \tag{3}$$

where θ_{12} is the leptonic angle, and θ_c the Cabibbo angle for quarks.

The phenomenological results that we obtain for the mixing angles only depend on the empirical pattern of neutral currents that we uncover in bare flavor space, and not on the size of the parameter characterizing the departure of the mixing matrix from unitarity (*i.e.*, in practice, the value of the counterterms [4]).

The latter, that need to be introduced to cancel unwanted non-diagonal transitions and to restore the standard CKM phenomenology [7], modify kinetic and mass terms of fermions. It turns out that the diagonalization of the new quadratic Lagrangian (kinetic + mass terms) obtained from the classical one by their adjunction requires non-unitary mixing matrices similar to the ones used in [2][3] to connect renormalized mass eigenstates to bare flavor eigenstates. The difference with respect to unitary matrices is proportional to Shabalin's kinetic counterterms, and, thus, depends on wave function renormalization(s). Nevertheless, we show, and the $SU(2)_L$ gauge symmetry plays a crucial role for this, that the mixing matrix occurring in charged currents of renormalized mass states, for example the Cabibbo matrix, stays unitary. In this (non-orthonormal) basis, the $SU(2)_L$ gauge algebra closes on the unit matrix which controls neutral currents (like it did in the orthonormal basis of bare mass eigenstates). Mixing angles simply undergo a renormalization depending on kinetic counterterms. By introducing a non-unitary renormalization of flavor states, one can also make unitary the mixing matrices which connect, in each sector, the renormalized flavor states to renormalized mass states; the former do not form either, however, an orthonormal basis.

The above results have been obtained, so far, without connection to horizontal symmetries; they only rely on the generalization to three generations of the empirical property concerning gauge neutral currents in flavor space, that we uncovered in [2][3] for two generations of quarks. This constitutes a departure from customary approaches, which rather try to induce some specific form for mass matrices from suitably guessed horizontal symmetries [8]. So, the last part of this work starts spanning a bridge between gauge currents and mass matrices, investigate which role is eventually played by flavor symmetries, and how they are realized in nature. For the sake of simplicity, we do this in the case of two generations only. A natural horizontal group arises, which is $SU(2)_f \times U(1)_f$ (or $U(2)_f$); the expressions of non-trivial parts of gauge neutral currents and of the fermion mass matrix (that we suppose to be real symmetric) respectively involve the $SU(2)_f(\theta)$ generators $\mathcal{T}_z(\theta)$ and $\mathcal{T}_x(\theta)$. It is a rotated version of the most trivial one (the generators of which are the Pauli matrices); its orientation depends on the mixing angle θ. It is unbroken in the case of mass degeneracy (and the mixing angle is then arbitrary); mass splittings alter this situation, and one can then find two subgroups leaving respectively invariant the gauge Lagrangian of neutral currents, or the fermionic mass terms (but not both). Mixing angles, associated, as we saw, to specific departure from unity of the matrix controlling neutral currents in flavor space, are accordingly also related to a specific pattern of the *breaking* of this $SU(2)_f$ [4]. We show that 2-dimensional flavor rotations, which are the transformations generated by the (θ independent) generator $\mathcal{T}_y$, continuously transform gauge neutral currents into the mass matrix.

Since introducing a unique constant mass matrix is known to be problematic in QFT when dealing with coupled systems [10], we then establish, through the $U(1)_{em}$ Ward identity, a connection between the photon-fermion-antifermion vertex and the fermionic self-energy. The same matrix as for other gauge

[4]That the breaking pattern of some underlying symmetry exhibits specific structures is not new since this kind of consideration is at the origin of mass relations among mesons or baryons in Gell-Mann's flavor $SU(3)$ (see for example [9] p.285).

neutral currents controls, inside the electromagnetic current, the violation of universality and FCNC's which occur in bare flavor space. Imposing that both sides of the Ward Identity are invariant by the flavor transformation that leaves the vertex invariant set constraints on the self-energy that we propose instead of "textures" because they stay, unlike the latter, invariant by flavor rotations.

Another important aspect of unitary flavor transformations is that, though they may not be symmetries of the theory (in the sense that its Lagrangian is not invariant), they should not change the "physics", in particular the Cabibbo angle occurring in charged gauge currents. We show that it is indeed the case, including its renormalization through the counterterms of Shabalin. Among these unitary transformations, flavor rotations turn out again to be of special interest. While they do not alter the breaking pattern (flavor group structure) of neutral currents in each sector ((u,c) and (d,s)), it is in general not the case for charged currents unless the rotations in the two sectors are identical. When it is so, only one of the two mixing angles (the one of (u,c) *or* the one of (d,s)) can be turned to zero, such that the one in the other sector becomes, as commonly assumed, equal to the Cabibbo angle. Flavor rotations appear as a very mildly broken symmetry of the Standard Model, in the sense that they only alter the Lagrangian through unphysical phase shifts and do not modify the "physics" (the Cabibbo mixing angle or its leptonic equivalent, masses ...).

The paper ends with various remarks and questions. Comparison with previous works is also done. The important issue of the alignment of mass and flavor states is investigated; that it can only occur in one of the two sectors is put in connection with the group structure of gauge charged currents; the empirical properties of mixing angles that have been uncovered inside neutral currents then naturally translate to the physical angles observed in the former. Unfortunately, we have in particular not been able to put the apparent quantization on the $\tan$ of twice the mixing angles as $n/2, n \in \mathbb{Z}$ in relation with the $SU(2)_f \times U(1)_f$ flavor group of symmetry that underlies electroweak physics for two flavors. The connection of the $\tan$ of the Cabibbo angle with the Golden ratio [3][11] stays a mystery the realm of which probably lies beyond the Standard Model.

2 Perturbative considerations

In this section, we show how 1-loop counterterms introduced by Shabalin [4] in order to cancel on mass-shell non-diagonal transitions between quark mass eigenstates entail, that mixing matrices linking (orthonormal) bare flavor states to renormalized mass states are in general non-unitary. This result is obtained by diagonalizing the whole quadratic (kinetic + mass) renormalized Lagrangian + counterterms. Kinetic counterterms (wave function renormalization) are shown to drive this non-unitarity. Accordingly, renormalized mass states do not form an orthonormal basis (as demonstrated in section 3 from basic QFT argumentation). Neutral currents being controlled in (both bare and renormalized) mass space, by the unit matrix (which we demonstrate), we exhibit the non-unit matrix which controls them, at 1-loop, in bare flavor space. We also show, by explicit calculations in the case of two generations, how $SU(2)_L$ gauge invariance preserves the unitarity of the Cabibbo matrix $\mathfrak{C}$ occurring in charged currents of renormalized mass eigenstates. It does not write anymore, however, as the product of the two renormalized mixing matrices occurring in bare neutral currents. We also show that, at the price of an additional non-unitary renormalization of bare flavor states, which then become non-orthonormal, too, one can go to unitary mixing matrices $\mathfrak{C}_{u,d}$ connecting, in each sector, renormalized mass states to renormalized flavor states. The standard relation $\mathfrak{C} = \mathfrak{C}_u^\dagger \mathfrak{C}_d$ is then restored.

2.1 The 1-loop self-energy

The study of neutral kaons [5] has unambiguously shown that, while flavor eigenstates can be assumed to form an orthonormal basis, mass eigenstates (K_{Long}, K_{Short}) do not (see footnote 2); the corresponding mixing matrix can only be non-unitary.

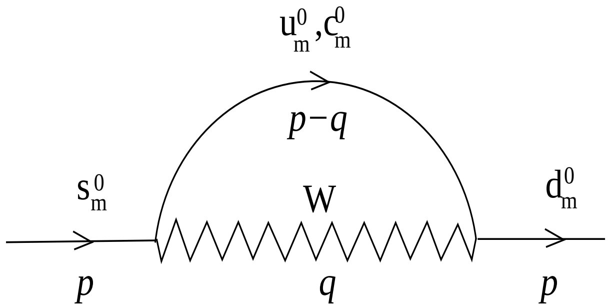

Fig. 1: $s^0_m \to d^0_m$ transitions at 1-loop

The situation could look very similar in the fermionic case, since there exist, for example, transitions between s^0_m and d^0_m [5] , depicted in Fig. 1. They have the form of a non-diagonal kinetic term (see subsection 2.1.1 for renormalization)

$$f_d(p^2, m_u^2, m_c^2, m_W^2)\ \bar d^0_m\, \not p(1-\gamma^5)\, s^0_m, \tag{4}$$

in which the function f_d is dimensionless and includes the factors $g^2 \sin\theta_c \cos\theta_c\ (m_c^2 - m_u^2)$ (θ_c is the classical Cabibbo angle). One should however also take into consideration the work [4] [6] which shows how the introduction of counterterms can make these transitions vanish for s^0_m *or* d^0_m on mass-shell [7]. The following non-diagonal counterterms, which are of two types, kinetic as well as mass terms, and with both chiral structures:

$$-A_d\ \bar d^0_m\, \not p(1-\gamma^5)\, s^0_m - B_d\ \bar d^0_m(1-\gamma^5)s^0_m - E_d\ \bar d^0_m\, \not p(1+\gamma^5)\, s^0_m - D_d\ \bar d^0_m(1+\gamma^5)s^0_m, \tag{5}$$

with

$$A_d = \frac{m_d^2\, f_d(p^2 = m_d^2) - m_s^2\, f_d(p^2 = m_s^2)}{m_d^2 - m_s^2}, \qquad E_d = \frac{m_s m_d \left(f_d(p^2 = m_d^2) - f_d(p^2 = m_s^2)\right)}{m_d^2 - m_s^2},$$
$$B_d = -m_s\, E_d, \qquad D_d = -m_d\, E_d, \tag{6}$$

are easily seen (see Appendix A) to play this role.

The kinetic counterterms for d-type quarks write (the L and R subscripts meaning respectively, throughout the paper, "left" $(1-\gamma^5)$ and "right" $(1+\gamma^5)$)

$$-A_d \begin{pmatrix} \bar d^0_{mL} & \bar s^0_{mL} \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \not p \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} - E_d \begin{pmatrix} \bar d^0_{mR} & \bar s^0_{mR} \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \not p \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix}, \tag{7}$$

and the mass counterterms

$$-\begin{pmatrix} \bar d^0_{mL} & \bar s^0_{mL} \end{pmatrix} \begin{pmatrix} & D_d \\ B_d & \end{pmatrix} \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} - \begin{pmatrix} \bar d^0_{mR} & \bar s^0_{mR} \end{pmatrix} \begin{pmatrix} & B_d \\ D_d & \end{pmatrix} \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix}. \tag{8}$$

Instead of the customary perturbative treatment of such counterterms in the bare orthonormal mass basis, order by order in the coupling constant, which can be rather cumbersome in this case [8], we shall instead

[5] s^0_m and d^0_m are the classical mass states obtained after diagonalization of the classical mass matrix by a bi-unitary transformation. At the classical level, they form an orthonormal basis; however, at 1-loop, non-local $s^0_m \leftrightarrow d^0_m$ can occur.

[6] The introduction of these counterterms enabled to show that, in a left-handed theory, the electric dipole moment of the quarks vanished up to 2-loops. This resulted in a neutron electric dipole moment well below experimental limits [12].

[7] Both cannot be of course simultaneously on mass-shell.

[8] In particular, when neither d nor s is on mass shell, which starts occurring at 2-loops, their role does not restrict anymore to the cancellation of non-diagonal transitions. See also subsection 8.4.

consider and re-diagonalize the effective renormalized Lagrangian at 1-loop

$$
\begin{aligned}
\mathcal{L} \;=\; & \begin{pmatrix} \bar d^0_{mL} & \bar s^0_{mL} \end{pmatrix} \begin{pmatrix} 1 & -A_d \\ -A_d & 1 \end{pmatrix} \not p \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} + \begin{pmatrix} \bar d^0_{mR} & \bar s^0_{mR} \end{pmatrix} \begin{pmatrix} 1 & -E_d \\ -E_d & 1 \end{pmatrix} \not p \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} \\
& - \begin{pmatrix} \bar d^0_{mL} & \bar s^0_{mL} \end{pmatrix} \begin{pmatrix} m_d & D_d \\ B_d & m_s \end{pmatrix} \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} - \begin{pmatrix} \bar d^0_{mR} & \bar s^0_{mR} \end{pmatrix} \begin{pmatrix} m_d & B_d \\ D_d & m_s \end{pmatrix} \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix}. \qquad (9)
\end{aligned}
$$

The advantage of doing so is that a link can then easily be established with section 3 which uses the general QFT argumentation of [2][3] to get similar results. The diagonalization of the quadratic Lagrangian (9) (kinetic + mass terms) proceeds as follows.

- Find 2 matrices $\mathcal{V}_d$ and $\mathcal{U}_d$ such that, for the kinetic terms

$$
\mathcal{V}_d^\dagger \begin{pmatrix} 1 & -A_d \\ -A_d & 1 \end{pmatrix} \mathcal{V}_d = 1 = \mathcal{U}_d^\dagger \begin{pmatrix} 1 & -E_d \\ -E_d & 1 \end{pmatrix} \mathcal{U}_d\ ; \qquad (10)
$$

they then rewrite

$$
\begin{pmatrix} \bar d^0_{mL} & \bar s^0_{mL} \end{pmatrix} \not p\ (\mathcal{V}_d^\dagger)^{-1}\mathcal{V}_d^{-1} \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} + \begin{pmatrix} \bar d^0_{mR} & \bar s^0_{mR} \end{pmatrix} \not p\ (\mathcal{U}_d^\dagger)^{-1}\mathcal{U}_d^{-1} \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix}, \qquad (11)
$$

which leads to introducing the new states

$$
\chi_{dL} = \mathcal{V}_d^{-1} \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix} = \mathcal{V}_d^{-1}\mathcal{C}_{d0}^{-1} \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix},\ \chi_{dR} = \mathcal{U}_d^{-1} \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} = \mathcal{U}_d^{-1}\mathcal{H}_{d0}^{-1} \begin{pmatrix} d^0_{fR} \\ s^0_{fR} \end{pmatrix}, \qquad (12)
$$

where $\mathcal{C}_{d0}$ and $\mathcal{H}_{d0}$ are the two unitary matrices by which the classical mass matrix M_0 has been diagonalized into $diag(m_d, m_s)$ [9]; we take them as follows [10]:

$$
\mathcal{C}_{d0} = \mathcal{R}(-\theta_{dL}), \quad \mathcal{H}_{d0} = \mathcal{R}(-\theta_{dR}), \qquad (13)
$$

where we have introduced the notation

$$
\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \qquad (14)
$$

Solutions to the conditions (10) are the *non-unitary* matrices depending respectively of arbitrary angles φ_{Ld} and φ_{Rd} [11] and arbitrary parameters ρ_d and σ_d:

---

[9] $diag(m_d, m_s) = \mathcal{C}_{d0}^\dagger M_0 \mathcal{H}_{d0}$, where M_0 is the classical mass matrix.

[10] We take a rotation matrix with angle $(-\theta_{dL})$ to match the formulæ of [2] [3].

[11] Maximal mixing, for example $\dfrac{1}{\sqrt 2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, also diagonalizes the kinetic terms, but into $\begin{pmatrix} 1+A_d & \\ & 1-A_d \end{pmatrix}$, which is not the canonical form (= the unit matrix). This accordingly requires two different renormalizations of the corresponding eigenvectors, which are finally $\sqrt{\frac{1+A_d}{2}}(d^0_m - s^0_m)$ and $\sqrt{\frac{1-A_d}{2}}(d^0_m + s^0_m)$. The mixing matrix connecting bare mass states to them is $\mathcal{V}_d = \frac{1}{\sqrt 2}\begin{pmatrix} \frac{1}{\sqrt{1+A_d}} & \frac{1}{\sqrt{1-A_d}} \\ -\frac{1}{\sqrt{1+A_d}} & \frac{1}{\sqrt{1-A_d}} \end{pmatrix}$, which is non-unitary (and non normal): it satisfies $\mathcal{V}_d^\dagger \mathcal{V}_d = \begin{pmatrix} \frac{1}{1+A_d} & \\ & \frac{1}{1-A_d} \end{pmatrix}$ and $\mathcal{V}_d \mathcal{V}_d^\dagger = \begin{pmatrix} 1 & A_d \\ A_d & 1 \end{pmatrix}$. This is why we look for general non-unitary $\mathcal{V}_d$ and $\mathcal{U}_d$. The special case outlined here corresponds to $\rho_d = 0$ and $\varphi_{Ld} = \pi/4$ in (15).

$$\begin{aligned}
\mathcal{V}_d &\overset{A_d\ small}{\approx} \mathcal{R}(\varphi_{Ld}) + A_d \begin{pmatrix} \dfrac{\rho_d-1}{2}\sin(\varphi_{Ld}) & -\dfrac{\rho_d-1}{2}\cos(\varphi_{Ld}) \\ \dfrac{\rho_d+1}{2}\cos(\varphi_{Ld}) & \dfrac{\rho_d+1}{2}\sin(\varphi_{Ld}) \end{pmatrix} \\
&= \mathcal{R}(\varphi_{Ld})\Big[1 - A_d\big(\mathcal{T}_z(\varphi_{Ld}) + i\rho_d\mathcal{T}_y\big)\Big], \\
\mathcal{U}_d &\overset{E_d\ small}{\approx} \mathcal{R}(\varphi_{Rd}) + E_d \begin{pmatrix} \dfrac{\sigma_d-1}{2}\sin(\varphi_{Rd}) & -\dfrac{\sigma_d-1}{2}\cos(\varphi_{Rd}) \\ \dfrac{\sigma_d+1}{2}\cos(\varphi_{Rd}) & \dfrac{\sigma_d+1}{2}\sin(\varphi_{Rd}) \end{pmatrix} \\
&= \mathcal{R}(\varphi_{Rd})\Big[1 - E_d\big(\mathcal{T}_z(\varphi_{Rd}) + i\sigma_d\mathcal{T}_y\big)\Big],
\end{aligned} \tag{15}$$

where we have introduced the notations

$$\mathcal{T}_z(\theta) = \frac{1}{2}\begin{pmatrix} \sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & -\sin 2\theta \end{pmatrix}, \quad \mathcal{T}_y = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \tag{16}$$

which will be often used in section 7, together with the $\mathcal{T}_x(\theta)$ generator which closes the corresponding $SU(2)_f$ algebra..

The connection between the flavor states and the $\chi_{L,R}$ states that diagonalize the kinetic terms goes accordingly through the non-unitary mixing matrices $\mathcal{C}_{d0}\,\mathcal{V}_d$ and $\mathcal{H}_{d0}\,\mathcal{U}_d$. At this stage, one has already made the transition from an *orthonormal bare mass basis* (d_m^0, s_m^0) [12] to *non-orthonormal* χ bases; the next bi-unitary transformation (below) will not change this fact.

• Express the renormalized mass matrix in the new $\chi_{L,R}$ basis

$$\begin{pmatrix} \bar{d}^0_{mL} & \bar{s}^0_{mL} \end{pmatrix} \begin{pmatrix} m_d & D_d \\ B_d & m_s \end{pmatrix} \begin{pmatrix} d^0_{mR} \\ s^0_{mR} \end{pmatrix} = \overline{\chi_{dL}}\,\mathcal{M}\,\chi_{dR},\ \mathcal{M} = \mathcal{V}_d^\dagger \begin{pmatrix} m_d & D_d \\ B_d & m_s \end{pmatrix} \mathcal{U}_d \tag{17}$$

and diagonalize it by a second bi-unitary transformation

$$V_d^\dagger\,\mathcal{M}\,U_d = \begin{pmatrix} \mu_d & \\ & \mu_s \end{pmatrix} \tag{18}$$

which, since it is bi-unitary, leaves the kinetic terms unchanged. The new (renormalized) mass eigenstates are accordingly

$$\begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix} = V_d^{-1}\chi_{dL} = V_d^{-1}\mathcal{V}_d^{-1}\mathcal{C}_{d0}^{-1}\begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}, \begin{pmatrix} d_{mR} \\ s_{mR} \end{pmatrix} = U_d^{-1}\chi_{dR} = U_d^{-1}\mathcal{U}_d^{-1}\mathcal{H}_{d0}^{-1}\begin{pmatrix} d^0_{fR} \\ s^0_{fR} \end{pmatrix}, \tag{19}$$

which correspond to the non-unitary mixing matrices $\mathcal{C}_{d0}\mathcal{V}_d V_d$ and $\mathcal{H}_{d0}\,\mathcal{U}_d U_d$ respectively for left-handed and right-handed fermions. The renormalized mass bases are accordingly non-orthonormal (see footnote 12).

• Parametrizing $V_d = \mathcal{R}(\theta_{2Ld})$, one uses the arbitrariness of φ_{Ld} to choose $\varphi_{Ld} + \theta_{2Ld} = 0$, which cancels the influence of the mass counterterms B_d, D_d and gives:

$$\mathcal{V}_d V_d = \begin{pmatrix} 1 & \dfrac{1-\rho_d}{2}A_d \\ \dfrac{1+\rho_d}{2}A_d & 1 \end{pmatrix}, \tag{20}$$

[12] since (d_m^0, s_m^0) is obtained from the bare flavor basis, supposed to be orthonormal, by a unitary transformation.

The mixing matrix $\mathcal{C}_d \equiv \mathcal{C}_{d0}\mathcal{V}_d V_d$ connecting the bare flavor states to the renormalized mass eigenstates

$$\left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right) = \mathcal{C}_d \left(\begin{array}{c} d_{mL} \\ s_{mL} \end{array}\right) \tag{21}$$

can be written, after some manipulations (commutations)

$$\mathcal{C}_d = \Big[1 - A_d\big(\mathcal{T}_z(\theta_{dL}) + i\rho_d\mathcal{T}_y\big)\Big]\mathcal{R}(-\theta_{dL}) \quad \Leftrightarrow \quad \mathcal{C}_d^{-1} = \mathcal{R}(\theta_{dL})\Big[1 + A_d\big(\mathcal{T}_z(\theta_{dL}) + i\rho_d\mathcal{T}_y\big)\Big]. \tag{22}$$

It satisfies in particular

$$(\mathcal{C}_d^{-1})^\dagger\mathcal{C}_d^{-1} = 1 + 2A_d\,\mathcal{T}_z(\theta_{dL}). \tag{23}$$

Eq.(23) is specially relevant since, once neutral currents are controlled, as we show later, in the (non-orthonormal) mass basis ξ_{dL} by the unit matrix, $(\mathcal{C}_d^{-1})^\dagger\mathcal{C}_d^{-1}$ provides, after introducing Shabalin's 1-loop counterterms in mass space, the renormalized 1-loop Lagrangian for neutral currents in the bare flavor basis [13].

When the system becomes degenerate, the reasons to forbid $d^0_m \leftrightarrow s^0_m$ on mass-shell transitions disappear and Shabalin's counterterms are expected to vanish. There is then no more need to introduce any non-unitary mixing matrix. The same result can be reached by the general QFT arguments of [2] since one can always choose an orthonormal basis of degenerate mass eigenstates; any connection between them and the (supposedly orthonormal) bare flavor basis goes then through unitary mixing matrices.

2.1.1 Renormalization; finiteness of the counterterms

The function f_d which appears in (4), calculated in the unitary gauge for the W boson and dimensionally regularized, is proportional to [4]

$$g^2 \sin\theta_c\cos\theta_c(m_c^2 - m_u^2)\int_0^1 dx\left[\frac{2x(1-x)}{\Delta(p^2)} + \frac{p^2x^3(1-x)}{M_W^2\Delta(p^2)} + \frac{x+3x^2}{M_W^2\Delta(p^2)^{2-n/2}}\Gamma(2-n/2)\right]. \tag{24}$$

$n = 4-\epsilon$ is the dimension of space-time, $\Delta(p^2) = (1-x)M_W^2 + x\frac{m_u^2+m_c^2}{2} - x(1-x)p^2$, and Γ is the Gamma function $\Gamma(\epsilon/2) = 2/\epsilon - \gamma + \ldots$ where $\gamma \approx 0.5772\ldots$ is the Euler constant. In particular, it includes a pole $(1/\epsilon)$ term and finite terms

$$f_d \ni g^2\sin\theta_c\cos\theta_c(m_c^2 - m_u^2)\int_0^1 dx\left[\frac{x+3x^2}{M_W^2}(2/\epsilon - \gamma) + \ finite(x, p^2, M_W^2, m_c^2, m_u^2)\right]; \tag{25}$$

we have decomposed the latter into the one proportional to the Euler constant, independent of p^2, m_c^2, m_u^2, and "$finite$", which depends on them. The transition corresponding to Fig. 1 gets, after renormalization (for example in the MS or $\overline{MS}$ schemes), a finite value

$$f_d^R(p^2, m_u^2, m_c^2, m_W^2)\ \bar{s}^0_m\,\not{p}(1-\gamma^5)\,d^0_m. \tag{26}$$

The cancellation of the (now finite) $d^0_m \leftrightarrow s^0_m$ transitions for d or s on-shell can be obtained by introducing the finite counterterms A_d, B_d, C_d, D_d given in (6). The MS and $\overline{MS}$ schemes, which differ by the subtraction of a constant proportional to γ in the integral (25), lead to different values for A_d, but identical values for B_d, E_d, D_d. It is noticeable that (6), when considered for bare f_d, leads to infinite A_d but to finite B_d, E_d, D_d. Likewise, the combination

$$(m_c^2 - m_u^2)A_u - (m_s^2 - m_d^2)A_d, \tag{27}$$

[13] This is also valid for the electromagnetic current, which is one among the gauge neutral currents. Up to the electric charge, it is controlled in mass space by the unit matrix and by the combination $(\mathcal{C}_d^{-1})^\dagger\mathcal{C}_d^{-1}$ in bare flavor space.

proportional (see (6)) to $m_c^2 f_u(p^2=m_c^2) - m_u^2 f_u(p^2=m_u^2) - m_s^2 f_d(p^2=m_s^2) + m_d^2 f_d(p^2=m_d^2)$, is finite [14]. This property results from the independence of the pole term in (24) on the quark masses, but for the global factors $(m_c^2-m_u^2)$ for f_d and $(m_s^2-m_d^2)$ for f_u. The finiteness of (27) entails in particular that A_u and A_d cannot vanish simultaneously and, thus, that a non-unitary mixing matrix is always at work in, at least, one of the two fermionic sectors $(u,c\ldots)$ and $(d,s\ldots)$. Like the $B_{u,d}$, $E_{u,d}$ and $D_{u,d}$ counterterms, the combination (27), which does not depend on the Euler constant γ, has the same value in MS and $\overline{MS}$; indeed, the aforementioned properties of the pole term are shared by the one proportional to γ in (24).

The four bare "infinite" functions $f_d(m_d^2)$, $f_d(m_s^2)$, $f_u(m_u^2)$ and $f_u(m_c^2)$ involved in the expressions of the counterterms A_d, A_u, B_d and B_u (and hence also of E_d, D_d and E_u, D_u) satisfy accordingly three conditions, resp. $B_d = cst$, $B_u = cst$, (27)$=cst$. The left-over arbitrariness corresponds to the renormalization prescription for A_d [15] which fixes, for example, $m_s^2 f_d^R(m_s^2) - m_d^2 f_d^R(m_d^2)$ (see (6)). It also corresponds to a renormalization prescription for f_d. The most common choices are MS and $\overline{MS}$, which lead to the same values of $B_{u,d}, E_{u,d}, D_{u,d}$ and of the combination (27), but other choices are *a priori* conceivable, which are eventually closer to "physics" (see subsection 8.3 and footnote 39, where we comment about the alignment of mass and flavor states in the (u,c) sector in connection with flavor rotations), and which can lead to different values for $B_{u,d}, E_{u,d}, D_{u,d}$ and for the combination (27).

A few remarks are due concerning the cancellation of ultraviolet infinities leading to a finite $W \to q_1\bar q_2$ amplitude. That a renormalization of the CKM matrix is mandatory to cancel infinities between the $(scalar)q_1\bar q_2$ and $Wq_1\bar q_2$ sectors when mass splittings are present was first shown in [13] for the case of two generations, and then in [14] in the case of three. In the present work, which uses the unitary gauge like in the section 3 of [13], only finite mass renormalization is needed and the only infinite counterterms that occur are the $A_{d,u}$ (kinetic counterterms corresponding to wave function renormalization). They become finite by a renormalization of $f_{d,u}$ (see (24)), which also makes finite the $s(c)^0_m \to d(u)^0_m$ 1-loop self-energy diagrams; both have indeed the same dependence on momentum and chirality. Showing that ultraviolet divergences cancel between the $(scalar)q_1\bar q_2$ and $Wq_1\bar q_2$ sectors amounts accordingly to showing that this infinite wave function renormalization is enough to make the observable $Wq_1\bar q_2$ vertex finite at 1-loop. The insertion of non-diagonal self-masses on any of the external legs of a bare $Wq_1\bar q_2$ vertex gives a vanishing contribution because one of the two fermions attached to it is always on mass-shell (Shabalin's counterterms are built up for this). So, the looked for cancellations correspond to the standard property of infinities coming from wave function renormalization to combine with those arising from the proper vertices (see for example [13]) to make, after a suitable charge renormalization, the 1-loop $Wq_1\bar q_2$ amplitude finite.

2.1.2 Summary of the perturbative 1-loop procedure

Since the procedure to go from the bare Lagrangian to the effective renormalized Lagrangian at 1-loop in flavor space is, though simple, not completely trivial, we make a brief summary of it below:

* the bare flavor basis can be supposed to be orthonormal;
* the bare mass basis, obtained from the diagonalization of the bare mass matrix by (bi)-unitary transformations, is orthonormal, too;
* in this bare mass basis, there appear at 1-loop non-diagonal transitions, and also flavor changing neutral currents;
* counterterms are introduced in this basis to cancel non-diagonal on mass-shell transitions;
* they alter the matrix of kinetic terms, which, in the same basis, is no longer 1, and the mass matrix, which is no longer diagonal;
* putting back kinetic terms to the unit matrix requires non-unitary transformations; the new states χ so defined do not form anymore an orthonormal basis;
* the mass matrix, including the newly added counterterms, has to be re-expressed in the χ bases and

[14] f_u is defined by a formula analogous to (24), with the exchange $m_c \leftrightarrow m_s, m_u \leftrightarrow m_d$.
[15] or for A_u, but the two choices cannot be independent.

re-diagonalized by a bi-unitary transformation; this does not change anymore the kinetic terms;
* this last diagonalization defines the renormalized mass states, which are obtained from the bare flavor states by a product of three matrices, two being unitary and one non-unitary; they accordingly do not form an orthonormal basis (the same result is obtained in section 3 from general considerations of QFT). This is the counterpart of canceling, on mass-shell, through counterterms, the non-diagonal, non-local transitions that occurred between orthogonal bare mass states. The situation, after renormalization, is thus very similar to the one studied in [5] for neutral kaons;
* once the (non-unitary) mixing matrix $\mathcal{C}$ linking renormalized mass states to bare flavor states at 1-loop has been defined by this procedure, we will show in subsection 2.2.2 that, in the renormalized (non-orthonormal) mass basis, the renormalized Lagrangian at 1-loop for neutral currents is controlled by the unit matrix. This entails that the quantity $(\mathcal{C}^{-1})^\dagger\mathcal{C}^{-1}$ determines the same Lagrangian in the bare flavor basis (it differs from the unit matrix, its usual expression in the absence of Shabalin's counterterms).

2.2 Gauge currents and renormalized mixing matrices

2.2.1 $SU(2)_L$ gauge symmetry: how the renormalized Cabibbo matrix stays unitary

$SU(2)_L$ gauge invariance, through the expression of the covariant derivatives of the fermionic fields, requires that the same counterterms that occur for the kinetic terms should also occur inside the gauge couplings. Let us consider a kinetic fermionic term in its canonical form $\overline{\Psi}\overleftrightarrow{\partial}\Psi \equiv \frac12\big(\overline{\Psi}\partial\Psi - (\overline{\partial\Psi})\Psi\big)$, and call A the generic kinetic counterterm. In the kinetic term ∂ is accordingly replaced with $A\partial$ and, introducing the covariant $SU(2)_L$ derivative in the two terms of $\overline{\Psi}\overleftrightarrow{\partial}\Psi$ yields $\frac12\overline{\Psi}A(\partial - ig\vec{W}.\vec{T})\Psi - \frac12\Big(\overline{A(\partial - ig\vec{W}.\vec{T})\Psi}\Big)\Psi = \overline{\Psi}A\partial\Psi - \frac{ig}{2}\overline{\Psi}(A\vec{T}+\vec{T}A).\vec{W}\Psi$. Calling

$$A = \left(\begin{array}{cc|cc} 1 & -A_u & & \\ -A_u & 1 & & \\ \hline & & 1 & -A_d \\ & & -A_d & 1 \end{array}\right) \tag{28}$$

the matrix of counterterms, the Lagrangian in bare mass space must accordingly include

$$\mathcal{L} \ni \left(\begin{array}{cccc} \bar{u}^0_{mL} & \bar{c}^0_{mL} & \bar{d}^0_{mL} & \bar{s}^0_{mL} \end{array}\right)\left(A\partial\!\!\!/ - \frac{ig}{2}(A\vec{T}+\vec{T}A).\vec{W}_\mu)\gamma^u \ldots\right)\left(\begin{array}{c} u^0_{mL} \\ c^0_{mL} \\ d^0_{mL} \\ s^0_{mL} \end{array}\right). \tag{29}$$

It is hermitian and involves the (Cabibbo rotated) $SU(2)_L$ generators $\vec{T}$

$$T^3 = \frac12\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), T^+ = \left(\begin{array}{c|c} & \mathcal{C}_0 \\ \hline & \end{array}\right), T^- = \left(\begin{array}{c|c} & \\ \hline \mathcal{C}_0^\dagger & \end{array}\right); \tag{30}$$

$\mathcal{C}_0$ is the bare Cabibbo matrix

$$\mathcal{C}_0 = \mathcal{C}_{u0}^\dagger\mathcal{C}_{d0} = \mathcal{R}(\theta_c), \quad \theta_c = \theta_{uL} - \theta_{dL}, \tag{31}$$

$\mathcal{C}_{d0}$ being the classical unitary mixing matrix in (d,s) sector given by (13) and $\mathcal{C}_{u0}$ its equivalent in the (u,c) sector, with bare mixing angle $(-\theta_{uL})$.

The mixing matrix has become, in the basis of bare mass eigenstates:

$$\mathcal{C} = \frac{1}{2}\left[\begin{pmatrix} 1 & -A_u \\ -A_u & 1 \end{pmatrix}\mathcal{C}_0 + \mathcal{C}_0\begin{pmatrix} 1 & -A_d \\ -A_d & 1 \end{pmatrix}\right], \tag{32}$$

which is not unitary. However, going to the final basis of mass eigenstates

$$\begin{pmatrix} u_{mL} \\ c_{mL} \end{pmatrix} = V_u^{-1}\mathcal{V}_u^{-1}\begin{pmatrix} u^0_{mL} \\ c^0_{mL} \end{pmatrix}, \quad \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix} = V_d^{-1}\mathcal{V}_d^{-1}\begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix}, \tag{33}$$

it becomes

$$\begin{aligned} \mathfrak{C} &= \frac{1}{2}V_u^\dagger\mathcal{V}_u^\dagger\left[\begin{pmatrix} 1 & -A_u \\ -A_u & 1 \end{pmatrix}\mathcal{C}_0 + \mathcal{C}_0\begin{pmatrix} 1 & -A_d \\ -A_d & 1 \end{pmatrix}\right]\mathcal{V}_d V_d \\ &= \mathcal{C}_u^\dagger\mathcal{C}_d - \frac{1}{2}(\mathcal{V}_u V_u)^\dagger\left[A_u\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\mathcal{C}_0 + A_d\,\mathcal{C}_0\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right]\mathcal{V}_d V_d, \end{aligned} \tag{34}$$

where we have used the expression on the left of (22) for $\mathcal{C}_d$ and its equivalent for $\mathcal{C}_u$. Choosing, as we did before, $\varphi_{Lu} + \theta_{2Lu} = 0 = \varphi_{Ld} + \theta_{2Ld}$, and using (20) for $\mathcal{V}_d V_d$ and its equivalent for $\mathcal{V}_u V_u$, one gets finally:

$$\mathfrak{C} = \begin{pmatrix} \cos\theta_c + \dfrac{\rho_u A_u - \rho_d A_d}{2}\sin\theta_c & \sin\theta_c + \dfrac{\rho_u A_u - \rho_d A_d}{2}\cos\theta_c \\ -\sin\theta_c + \dfrac{\rho_u A_u - \rho_d A_d}{2}\cos\theta_c & \cos\theta_c + \dfrac{\rho_u A_u - \rho_d A_d}{2}\sin\theta_c \end{pmatrix} \approx \mathcal{R}\left(\theta_c - \frac{\rho_u A_u - \rho_d A_d}{2}\right). \tag{35}$$

So, once Shabalin's counterterms and the change of basis have both been taken into account, the renormalized Cabibbo matrix, which does not write anymore as the product $\mathcal{C}_u^\dagger\mathcal{C}_d$, becomes again unitary [16]. This occurs because of $SU(2)_L$ gauge invariance, and despite the fact that neither $\mathcal{C}_u$ nor $\mathcal{C}_d$ is unitary. With respect to its classical value, the classical Cabibbo angle $\theta_c = \theta_{uL} - \theta_{dL}$ gets renormalized by $\dfrac{\rho_u A_u - \rho_d A_d}{2}$.

2.2.2 Neutral currents and the closure of the $SU(2)_L$ algebra

Like charged currents, the form of neutral currents is determined by gauge invariance, through the $SU(2)_L$ covariant derivative. It is given in the bare mass basis by (29), which easily translates to the renormalized mass basis since the latter deduces from the former by the transformations (33).

The procedure is specially simple since the T^3 generator only involves unit matrices in each sector, such that $V^{-1}\mathcal{V}^{-1}$ can freely move through it. It is furthermore easy to check that, in addition to (10), one has

$$(\mathcal{V}_{u,d}V_{u,d})^\dagger\begin{pmatrix} 1 & -A_{u,d} \\ -A_{u,d} & 1 \end{pmatrix}\mathcal{V}_{u,d}V_{u,d} = 1, \tag{36}$$

such that the 1-loop effective Lagrangian for neutral currents gets controlled by the unit matrix in the renormalized mass basis.

[16] The customary expression $\mathcal{C}_u^\dagger\mathcal{C}_d$ for the CKM matrix is not unitary and should be discarded. One gets indeed, with straightforward notations, $\mathcal{C}^\dagger\mathcal{C} \approx 1 - 2\mathcal{R}(\theta_{dL})\Big(A_u\mathcal{T}_z(\theta_{uL}) + A_d\mathcal{T}_z(\theta_{dL})\Big)\mathcal{R}(-\theta_{dL})$

So, $SU(2)_L$ gauge invariance ensures that neutral currents are controlled by the unit matrix:
- at the classical level in the basis of bare (orthonormal) mass states;
- in the Lagrangian renormalized at 1-loop in the basis of renormalized (non-orthonormal) mass sates.

After Shabalin's counterterms $A_u \equiv \epsilon_u$ and $A_d \equiv \epsilon_d$ have been included, in the renormalized mass bases the $SU(2)_L$ generators write

$$T^3 = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), T^+ = \left(\begin{array}{c|c} & \mathfrak{C} \\ \hline & \end{array}\right), T^- = \left(\begin{array}{c|c} & \\ \hline \mathfrak{C}^\dagger & \end{array}\right); \tag{37}$$

of course, the unitarity of $\mathfrak{C}$ is necessary for its closure on the unit matrix in the neutral gauge sector.

2.2.3 Charged gauge currents in flavor space; renormalized flavor states

It is now interesting to write back the renormalized Lagrangian in bare flavor space (it is in this basis that we uncovered empirical specific breaking patterns). One starts from (32) in bare mass space and go to bare flavor space by the bare mixing matrices $\mathcal{C}_{u0}$ and $\mathcal{C}_{d0}$; this yields

$$\begin{aligned}
&\left(\begin{array}{cc} \bar u_{mL} & \bar c_{mL} \end{array}\right) \mathfrak{C}\gamma^\mu \left(\begin{array}{c} d_{mL} \\ s_{mL} \end{array}\right) \\
&= \left(\begin{array}{cc} \bar u^0_{fL} & \bar c^0_{fL} \end{array}\right) \frac{1}{2}\left[\mathcal{C}_{u0}\left(\begin{array}{cc} 1 & -A_u \\ -A_u & 1 \end{array}\right)\mathcal{C}^\dagger_{u0} + \mathcal{C}_{d0}\left(\begin{array}{cc} 1 & -A_d \\ -A_d & 1 \end{array}\right)\mathcal{C}^\dagger_{d0}\right]\gamma^\mu \left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right) \\
&= \left(\begin{array}{cc} \bar u^0_{fL} & \bar c^0_{fL} \end{array}\right) \left[1 + A_u \mathcal{T}_z(\theta_{uL}) + A_d \mathcal{T}_z(\theta_{dL})\right]\gamma^\mu \left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right) \qquad (38) \\
&= \left(\begin{array}{cc} \bar u^0_{fL} & \bar c^0_{fL} \end{array}\right) \left[1 + A_u \mathcal{T}_z(\theta_{uL})\right]\left[1 + A_d \mathcal{T}_z(\theta_{dL})\right]\gamma^\mu \left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right) \\
&\approx \overline{e^{A_u \mathcal{T}_z(\theta_{uL})}\left(\begin{array}{c} u^0_{fL} \\ c^0_{fL} \end{array}\right)}\gamma^\mu e^{A_d \mathcal{T}_z(\theta_{dL})}\left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right), \qquad (39)
\end{aligned}$$

where we have used the expression for $\mathcal{T}_z(\theta)$ given in (16) and the relations $\mathcal{C}_{u0,d0}\left(\begin{array}{cc} & 1 \\ 1 & \end{array}\right)\mathcal{C}^\dagger_{u0,d0} = -2\mathcal{T}_z(\theta_{uL,dL})$. It is therefore possible to define as "renormalized flavor states" the ones that appear in the last line of (39)

$$\left(\begin{array}{c} d_{fL} \\ s_{fL} \end{array}\right) = e^{A_d \mathcal{T}_z(\theta_{dL})}\left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right) \text{ and } \left(\begin{array}{c} u_{fL} \\ c_{fL} \end{array}\right) = e^{A_u \mathcal{T}_z(\theta_{uL})}\left(\begin{array}{c} u^0_{fL} \\ c^0_{fL} \end{array}\right). \tag{40}$$

They are deduced from the bare flavor states by the non-unitary transformations $e^{A_{u,d}\mathcal{T}_z(\theta_{uL,dL})}$, and do not form anymore, accordingly, an orthonormal basis. In the renormalized flavor basis, the $SU(2)_L$ generators write in their simplest form

$$T^3 = \frac{1}{2}\left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right), T^+ = \left(\begin{array}{c|c} & 1 \\ \hline & \end{array}\right), T^- = \left(\begin{array}{c|c} & \\ \hline 1 & \end{array}\right); \tag{41}$$

universality is thus achieved together with the absence of FCNC's, like in the basis or renormalized mass states. The two points of view describe of course the same physics: in the non-orthonormal renormalized flavor basis, neutral currents are controlled by the unit matrix (seemingly absence of non-diagonal transitions, but they still occur through the non-orthogonality of the states), and, in the bare flavor basis, neutral currents are controlled by a matrix slightly different from the unit matrix (non-diagonal transitions between orthogonal states are then conspicuous).

The last step is to calculate the mixing matrices $\mathfrak{C}_{u,d}$ linking the renormalized mass states (see (19)) to the renormalized flavor states $f_{uL,dL}$ defined in (40). From (22), it is straightforward to deduce

$$\begin{aligned}\mathfrak{C}_d &= e^{A_d T_z(\theta_{dL})}\mathcal{C}_d \approx (1 + A_d T_z(\theta_{dL}))\,\mathcal{C}_d \\ &\stackrel{(22)}{=} (1 + A_d T_z(\theta_{dL}))\,[1 - A_d\,(\mathcal{T}_z(\theta_{dL}) - i\rho_d \mathcal{T}_y)]\,\mathcal{R}(-\theta_{dL}) \\ &= \Big(1 + i\rho_d A_d \mathcal{T}_y\Big)\mathcal{R}(-\theta_{dL}) \approx \mathcal{R}(-\theta_{dL} - \frac{\rho_d A_d}{2}).\end{aligned} \tag{42}$$

which is unitary. The relation

$$\mathfrak{C} = \mathfrak{C}_u^\dagger \mathfrak{C}_d, \tag{43}$$

is seen to be now restored. So, renormalized mass states are connected to renormalized flavor states through unitary mixing matrices. In the renormalized flavor basis, the sole effects of Shabalin's counterterms is a renormalization of the mixing angles.

After all these steps have been gone through, the 1-loop renormalized Lagrangian writes identically to the bare Lagrangian with:
$*$ renormalized masses;
$*$ renormalized, non-orthonormal (mass and flavor) eigenstates;
$*$ unitary mixing matrices with renormalized mixing angles.

It has the same form as the bare Lagrangian of the Standard Model, except that the notion of flavor has been redefined, such that it no longer appears as a strictly conserved quantity.

3 Neutral currents of bare flavor eigenstates; general QFT argumentation

After establishing by perturbative arguments the *a priori* non-unitarity of mixing matrices for non-degenerate coupled systems, we come back to the argumentation of [2] based on general principles of Quantum Field Theory, then generalize it to the case of three generations. Unlike in the previous section, the argumentation goes beyond perturbation theory. There, for example, the two mixing angles which could be introduced *de facto* in $\mathcal{V}$ (see (15)), arose through perturbative arguments and were perturbatively close to each other; we call them Cabibbo-like. At the opposite, "maximal mixing" solutions of the "unitarization equations" (see subsection 4 below), which occur in addition to Cabibbo-like solutions, form a discrete set of solutions superimposed to the former, and arise independently of perturbative arguments. The property of maximal mixing to be non-perturbative is in agreement with its common association with quasi-degenerate systems (the smaller the mixing angle, the bigger the mass hierarchy [15]), for which small variations (for example in the mass spectrum) can have large effects on eigenstates, and thus on the mixing angles themselves.

The only "perturbative expansions" that will be performed (in sections 5 and 6) concern small deviations from the solutions of the "unitarization equations".

3.1 Different basis of fermions

Three bases generally occur in the treatment of fermions:
$*$ flavor eigenstates: (u_f, c_f, t_f) and (d_f, s_f, b_f) for quarks, (e_f, μ_f, τ_f) and $(\nu_{ef}, \nu_{\mu f}, \nu_{\tau f})$ for leptons;
$*$ mass eigenstates: (u_m, c_m, t_m) and (d_m, s_m, b_m) for quarks, (e_m, μ_m, τ_m) and $(\nu_{em}, \nu_{\mu m}, \nu_{\tau m})$ for leptons. They include in particular the charged leptons detected experimentally, since their identification

proceeds through the measurement of their $charge/mass$ ratio in a magnetic field; these eigenstates are the ones of the full renormalized propagator at its poles; at 1-loop, they can be identified with components of the renormalized mass states of (19) in section 2;

$*$ for leptons, one often invokes a third type of basis, made with the neutrino states that couple to the mass eigenstates of charged leptons in charged weak currents. These are the so-called ”electronic”, “muonic” and ”τ” neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ considered in SM textbooks: they are indeed identified by the outgoing charged leptons that they produce through charged weak currents, and the latter are precisely mass eigenstates (see above). They read

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = K_\ell^\dagger \begin{pmatrix} \nu_{ef} \\ \nu_{\mu f} \\ \nu_{\tau f} \end{pmatrix} = (K_\ell^\dagger K_\nu) \begin{pmatrix} \nu_{em} \\ \nu_{\mu m} \\ \nu_{\tau m} \end{pmatrix}, \tag{44}$$

where K_ℓ and K_ν are the mixing matrices respectively of charged leptons and of neutrinos (*i.e.* the matrices that connect their flavor to their mass eigenstates). These neutrinos are neither flavor nor mass eigenstates; they coincide with the latter when the mixing matrix *of charged leptons* is taken equal to unity $K_\ell = 1$, *i.e.* when the mass and flavor eigenstates of charged leptons are aligned, which is often assumed in the literature.

3.2 Mixing matrices. Notations

We start again with the case of two generations, and use the notations of [2]. The situation is depicted on Fig. 2 [17]. The bare flavor states, independent of $q^2 = z$, are ψ_1 and ψ_2 (they can be for example the d^0_{fL} and s^0_{fL} of section 2) and we suppose that they are orthonormal. Three orthonormal bases, respectively made of a pair of eigenvectors of the (hermitian) renormalized quadratic Lagrangian at three different values of z, can be seen. The first corresponds to the physical mass $z = z_1 = m_1^2$; the second, made of $\psi^1(z)$ and $\psi^2(z)$, corresponds to an arbitrary z; the last corresponds to the second physical mass $z = z_2 = m_2^2$. Within the first basis one finds the first physical mass eigenstate, ϕ^1_m, and a second (non-physical) eigenstate, ω_1^2; the third basis is made of the second physical mass eigenstate, ϕ^2_m, and of a second (non-physical) eigenstate, ω_2^1 [18]. For example, at 1-loop, ϕ^1_m and ϕ^2_m can be identified with the two components of $\begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}$ (see (19) in section 2).

[17]This figure was already published in [2]. Its inclusion in the present work makes it more easily understandable and self-contained.

[18]On Fig. 2, the $\lambda(z)$'s are the eigenvalues of the inverse renormalized propagator at $z = q^2$.

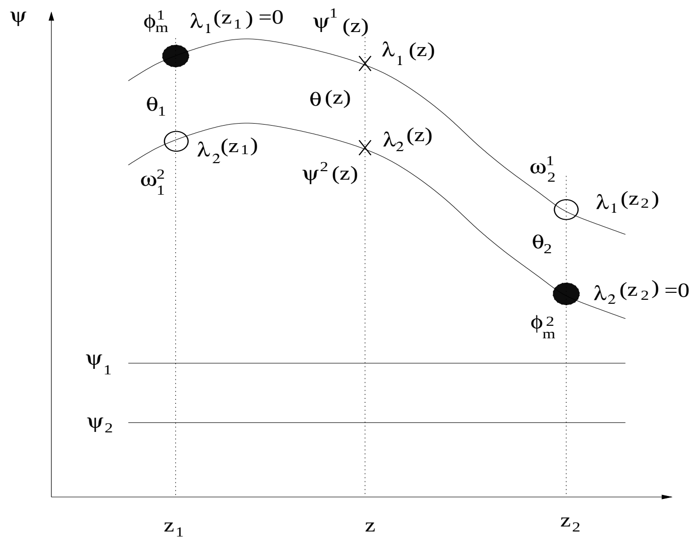

Fig. 2: Eigenstates of a binary complex system

The flavor states ψ_1 and ψ_2 can be expressed in both orthonormal bases (ϕ^1_m, ω^2_1) and (ϕ^2_m, ω^1_2) according to

$$\begin{aligned} \psi_1 &= c_1\phi^1_m - s_1\omega^2_1 = c_2\omega^1_2 - s_2\phi^2_m, \\ \psi_2 &= s_1\phi^1_m + c_1\omega^2_1 = s_2\omega^1_2 + c_2\phi^2_m, \end{aligned} \tag{45}$$

which yields

$$\begin{pmatrix} \phi^1_m \\ \phi^2_m \end{pmatrix} = \begin{pmatrix} c_1 & s_1 \\ -s_2 & c_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{46a}$$

$$\Leftrightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{c_1c_2 + s_1s_2} \begin{pmatrix} c_2 & -s_1 \\ s_2 & c_1 \end{pmatrix} \begin{pmatrix} \phi^1_m \\ \phi^2_m \end{pmatrix}. \tag{46b}$$

Since ψ_1 and ψ_2 have been assumed to form an orthonormal basis, eq.(46a) entails

$$|\phi^1_m| = 1 = |\phi^2_m|, \; < \phi^2_m \mid \phi^1_m >= s_1c_2 - c_1s_2 \stackrel{\theta_2\neq\theta_1}{\neq} 0. \tag{47}$$

(46a) shows that, for two generations, the mixing matrix $\mathcal{C}$ satisfies [19]

$$\mathcal{C}^{-1} = \begin{pmatrix} c_1 & s_1 \\ -s_2 & c_2 \end{pmatrix}. \tag{48}$$

We generalize this, in the following, to the case of three generations by writing the corresponding mixing matrix K^{-1} as a product of three matrices, which reduce, in the unitarity limit, to the basic rotations by $-\theta_{12}$, $-\theta_{23}$ and $-\theta_{13}$ (we are not concerned with CP violation)

$$K^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -\tilde{s}_{13} & 0 & \tilde{c}_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

[19] This corresponds to $\rho_d = 1$ in formula (22).

We parametrize each basic matrix, which is *a priori* non-unitary, with two angles, respectively $(\theta_{12}, \tilde\theta_{12})$, $(\theta_{23}, \tilde\theta_{23})$ and $(\theta_{13}, \tilde\theta_{13})$ [20]. We deal accordingly with six mixing angles, instead of three in the unitary case (where $\tilde\theta_{ij} = \theta_{ij}$). We will use throughout the paper the notations $s_{ij} = \sin(\theta_{ij}), \tilde s_{ij} = \sin(\tilde\theta_{ij})$, and likewise, for the cosines, $c_{ij} = \cos(\theta_{ij}), \tilde c_{ij} = \cos(\tilde\theta_{ij})$.

To lighten the text, the elements of $(K^{-1})^\dagger K^{-1}$ will be abbreviated by $[ij], i,j = 1 \ldots 3$ instead of $((K^{-1})^\dagger K^{-1})_{[ij]}$, and the corresponding neutral current will be noted $\{ij\}$. So, in the quark case, $\{12\}$ stands for $\bar u_f \gamma^\mu_L c_f$ or $\bar d_f \gamma^\mu_L s_f$, and, in the neutrino case, for $\bar\nu_{ef}\gamma^\mu_L \nu_{\mu f}$ or $\bar e_f \gamma^\mu_L \mu_f$.

4 The unitary approximation

In a first approximation, mixing matrices are unitary, such that neutral currents are very close to being controlled in the renormalized mass basis, too, by the unit matrix. The corresponding equations (unitarization conditions) will determine the equivalent of "classical solutions", away from which we shall then consider small deviations which exist because of mass splittings: non-degeneracy generates a tiny departure from unitarity of the corresponding mixing matrices and, accordingly, a tiny departure from unity of the matrix controlling neutral currents in bare flavor space.

The unitarization conditions simply express the absence of non-diagonal neutral currents in flavor space, and universality for their diagonal counterparts, assuming that the gauge Lagrangian of neutral currents is controlled in mass space by the unit matrix; they accordingly summarize into

$$(K^{-1})^\dagger K^{-1} = 1. \tag{50}$$

There are five equations: three arise from the absence of non-diagonal neutral currents, and two from the universality of diagonal currents. Accordingly, one degree of freedom is expected to be unconstrained.

4.1 Absence of non-diagonal neutral currents of flavor eigenstates

The three conditions read:

$*$ for the absence of $\{13\}$ and $\{31\}$ currents:

$$[13] = 0 = [31] \Leftrightarrow c_{12}\left[c_{13}s_{13} - \tilde c_{13}\tilde s_{13}(\tilde c_{23}^2 + s_{23}^2)\right] - \tilde c_{13}\tilde s_{12}(c_{23}s_{23} - \tilde c_{23}\tilde s_{23}) = 0; \tag{51}$$

$*$ for the absence of $\{23\}$ and $\{32\}$ currents:

$$[23] = 0 = [32] \Leftrightarrow s_{12}\left[c_{13}s_{13} - \tilde c_{13}\tilde s_{13}(\tilde c_{23}^2 + s_{23}^2)\right] + \tilde c_{13}\tilde c_{12}(c_{23}s_{23} - \tilde c_{23}\tilde s_{23}) = 0; \tag{52}$$

$*$ for the absence of $\{12\}$ and $\{21\}$ currents:

$$\begin{aligned}&[12] = 0 = [21] \Leftrightarrow \\ &s_{12}c_{12}c_{13}^2 - \tilde s_{12}\tilde c_{12}(c_{23}^2 + \tilde s_{23}^2) + s_{12}c_{12}\tilde s_{13}^2(s_{23}^2 + \tilde c_{23}^2) + \tilde s_{13}(s_{12}\tilde s_{12} - c_{12}\tilde c_{12})(c_{23}s_{23} - \tilde c_{23}\tilde s_{23}) = 0.\end{aligned} \tag{53}$$

[20]So doing, we do not consider the most general non-unitary mixing matrices. All possible phases were included in [2][3], where they have been shown to finally, in the case of two generations, drop out of the final results. There is another reason to ignore them here, specially in the case of three generations (in addition to the point that they would make the equations to solve extremely difficult to handle analytically): such phases can be expected to trigger CP violation, even in the case of two generations. We consider that the corresponding extensive study should be the subject of a separate work. CP violation is not our concern here.

4.2 Universality of diagonal neutral currents of flavor eigenstates

The two independent conditions read:

$*$ equality of $\{11\}$ and $\{22\}$ currents:

$$\begin{aligned}&[11]-[22]=0 \Leftrightarrow\\&(c_{12}^2-s_{12}^2)\left[c_{13}^2+\tilde s_{13}^2(s_{23}^2+\tilde c_{23}^2)\right]-(\tilde c_{12}^2-\tilde s_{12}^2)(c_{23}^2+\tilde s_{23}^2)\\&\qquad\qquad+2\tilde s_{13}(c_{23}s_{23}-\tilde c_{23}\tilde s_{23})(c_{12}\tilde s_{12}+s_{12}\tilde c_{12})=0;\end{aligned} \tag{54}$$

$*$ equality of $\{22\}$ and $\{33\}$ currents:

$$\begin{aligned}&[22]-[33]=0 \Leftrightarrow\\&s_{12}^2+\tilde c_{12}^2(c_{23}^2+\tilde s_{23}^2)-(s_{23}^2+\tilde c_{23}^2)+(1+s_{12}^2)\left[\tilde s_{13}^2(s_{23}^2+\tilde c_{23}^2)-s_{13}^2\right]\\&\qquad\qquad+2s_{12}\tilde s_{13}\tilde c_{12}(\tilde c_{23}\tilde s_{23}-c_{23}s_{23})=0.\end{aligned} \tag{55}$$

The equality of $\{11\}$ and $\{33\}$ currents is of course not an independent condition.

4.3 Solutions for $\theta_{13}=0=\tilde\theta_{13}$

In a first step, to ease solving the system of trigonometric equations, we shall study the configuration in which one of the two angles parametrizing the 1-3 mixing vanishes [21], which is very close to what is observed experimentally in the quark sector, and likely in the neutrino sector. It turns out, as demonstrated in Appendix C, that the second mixing angle vanishes simultaneously. We accordingly work in the approximation (the sensitivity of the solutions to a small variation of $\theta_{13},\tilde\theta_{13}$ will be studied afterwards)

$$\theta_{13}=0=\tilde\theta_{13}. \tag{56}$$

Eqs. (51), (52), (53), (54) and (55), reduce in this limit to

$$-\tilde s_{12}(c_{23}s_{23}-\tilde c_{23}\tilde s_{23})=0, \tag{57a}$$

$$\tilde c_{12}(c_{23}s_{23}-\tilde c_{23}\tilde s_{23})=0, \tag{57b}$$

$$s_{12}c_{12}-\tilde s_{12}\tilde c_{12}(c_{23}^2+\tilde s_{23}^2)=0, \tag{57c}$$

$$(c_{12}^2-s_{12}^2)-(\tilde c_{12}^2-\tilde s_{12}^2)(c_{23}^2+\tilde s_{23}^2)=0, \tag{57d}$$

$$s_{12}^2+\tilde c_{12}^2(c_{23}^2+\tilde s_{23}^2)-(s_{23}^2+\tilde c_{23}^2)=0. \tag{57e}$$

It is shown in Appendix D that the only solutions are:

$*$ $\tilde\theta_{23}=\theta_{23}+k\pi$ Cabibbo-like, associated with either $\theta_{12}=\tilde\theta_{12}+m\pi$ Cabibbo-like or θ_{12} and $\tilde\theta_{12}$ maximal;

$*$ $\tilde\theta_{12}=\theta_{12}+r\pi$ Cabibbo-like, associated with θ_{23} and $\tilde\theta_{23}$ maximal.

Accordingly, the two following sections will respectively start from:

$*$ θ_{12} and θ_{23} Cabibbo-like (and, in a first step, vanishing θ_{13}), which finally leads to a mixing pattern similar to what is observed for quarks;

$*$ θ_{23} maximal and θ_{12} Cabibbo like (and, in a first step, vanishing θ_{13}), which finally leads to a mixing pattern similar to the one observed for neutrinos.

[21]By doing so, we exploit the possibility to fix one degree of freedom left *a priori* unconstrained by the five equations; see subsection 7.

5 Beyond unitarity. The quark sector; constraining the CKM angles

Because of mass splittings, the "unitarization equations" of subsection 4 cannot be exactly satisfied. This is why, in the following, mixing matrices connecting bare flavor states to (renormalized) mass states are considered to only belong to the vicinity of the (unitary) solutions of these equations. Characterizing this departure from unitarity is the subject of this section and of the next one dealing with leptons. We show that all their mixing angles satisfy the straightforward generalization to three generations of the empirical criterion satisfied to a high precision, for two generations of quarks, by the Cabibbo angle [3]: for each pair of fermions of the same type, universality in the space of bare flavor states is verified with the same accuracy as the absence of FCNC's. It cannot be deduced, up to now, from general principles and stays an empirical property the origin of which should presumably be looked for "beyond the Standard Model" [22].

We accordingly investigate, in the following, the possibility that, in agreement with the reported criterion, the product $(K^{-1}_{u,d})^\dagger K^{-1}_{u,d}$, with K given by (49), be of the form

$$(K^{-1}_{u,d})^\dagger K^{-1}_{u,d} - 1 = \left(\begin{array}{ccc} \alpha_{u,d} & \pm(\alpha_{u,d}-\beta_{u,d}) & \pm(\alpha_{u,d}-\gamma_{u,d}) \\ \pm(\alpha_{u,d}-\beta_{u,d}) & \beta_{u,d} & \pm(\beta_{u,d}-\gamma_{u,d}) \\ \pm(\alpha_{u,d}-\gamma_{u,d}) & \pm(\beta_{u,d}-\gamma_{u,d}) & \gamma_{u,d} \end{array}\right); \tag{58}$$

so, as conspicuous on (58), any difference between two diagonal elements (for example [11] - [33]) is identical to $\pm$ the corresponding non-diagonal ones (in this case [13] and [31]). The resulting conditions yields a system of trigonometric equations for the six mixing angles $\theta_{12}, \tilde\theta_{12}, \theta_{23}, \tilde\theta_{23}, \theta_{13}$ and $\tilde\theta_{13}$. Without exhibiting the whole set of its solutions, we are able to show that it includes all measured values of fermionic mixing angles up to a precision smaller than the experimental uncertainty.

For the case of quarks, all mixing angles will be considered to belong to the neighborhood of Cabibbo-like solutions of the unitarization equations (this will be different for the case of leptons in section 6, where θ_{23} will be considered to belong to the neighborhood of the maximal mixing, also solution of these equations).

5.1 The simplified case $\theta_{13} = 0 = \tilde\theta_{13}$

In the neighborhood of the solution with both θ_{12} and θ_{23} Cabibbo-like, we write

$$\begin{array}{rcl} \tilde\theta_{12} &=& \theta_{12}+\epsilon, \\ \tilde\theta_{23} &=& \theta_{23}+\eta. \end{array} \tag{59}$$

The pattern ($\theta_{13} = 0 = \tilde\theta_{13}$) can be reasonably considered to be close to the experimental situation, at least close enough for trusting not only the relations involving the first and second generation, but also the third one.

Like in [3], we impose that the absence of $\{12\}, \{21\}$ neutral currents is violated with the same strength as the universality of $\{11\}$ and $\{22\}$ currents. (57c) and (57d) yield

$$|2\eta s_{12}c_{12}s_{23}c_{23} + \epsilon(c_{12}^2 - s_{12}^2)| = |-2\eta s_{23}c_{23}(c_{12}^2 - s_{12}^2) + 4\epsilon s_{12}c_{12}|. \tag{60}$$

We choose the "+" sign for both sides, such that, for two generations only, the Cabibbo angle satisfies $\tan(2\theta_{12}) = +1/2$. (60) yields the ratio η/ϵ, that we then plug into the condition equivalent to (60) for the $(2,3)$ channel, coming from (57b)(57e)

$$|\eta c_{12}(c_{23}^2 - s_{23}^2)| = |2\eta s_{23}c_{23}(1+c_{12}^2) - 2\epsilon s_{12}c_{12}|. \tag{61}$$

[22] Notice that it is satisfied a mixing matrix equal to the unit matrix (alignment of mass and flavor states) since universality and absence of FCNC's are both fulfilled; accordingly they both undergo identical (vanishing) violations.

(60) and (61) yield

$$\tan(2\theta_{23}) = \frac{c_{12}}{1 + c_{12}^2 - 2s_{12}c_{12}\dfrac{(s_{12}c_{12} + c_{12}^2 - s_{12}^2)}{4s_{12}c_{12} - (c_{12}^2 - s_{12}^2)}} \approx \frac{c_{12}}{2 - \dfrac{5}{4}\dfrac{s_{12}c_{12}}{\tan(2\theta_{12}) - \dfrac{1}{2}}}. \tag{62}$$

In the r.h.s. of (62), we have assumed that θ_{12} is close to its Cabibbo value $\tan(2\theta_{12}) \approx 1/2$. θ_{23} is seen to vanish with $[\tan(2\theta_{23}) - 1/2]$. The value obtained for θ_{23} is plotted in Fig. 3 as a function of θ_{12}, together with the experimental intervals for θ_{23} and θ_{12}. There are two [17] for θ_{12}; the first comes from the measures of V_{ud} (black (dark) vertical lines on Fig. 3)

$$V_{ud} \in [0.9735, 0.9740] \Rightarrow \theta_{12} \in [0.2285, 0.2307], \tag{63}$$

and the second from the measures of V_{us} (purple (light) vertical lines on Fig. 3)

$$V_{us} \in [0.2236, 0.2278] \Rightarrow \theta_{12} \in [0.2255, 0.2298]. \tag{64}$$

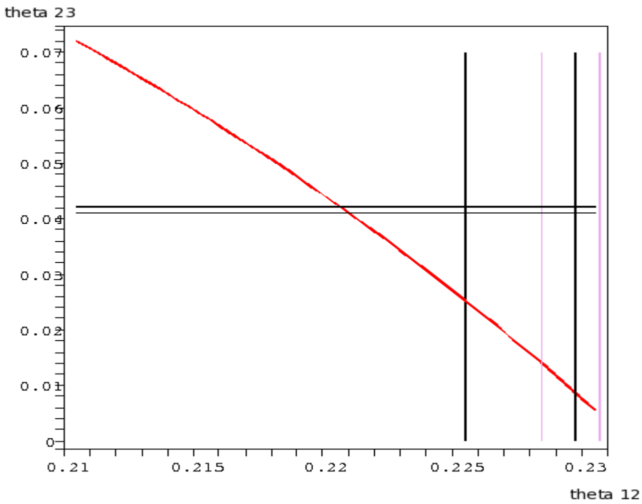

Fig. 3: θ_{23} for quarks as a function of θ_{12}; simplified case $\theta_{13} = 0 = \tilde\theta_{13}$

The measured value for θ_{23} is seen on Fig. 3 to correspond to $\theta_{12} \approx 0.221$, that is $\cos(\theta_{12}) \approx 0.9757$. The value that we get for $\cos(\theta_{12})$ is accordingly $1.7\,10^{-3}$ away from the upper limit of the present upper bound for $V_{ud} \equiv c_{12}c_{13}$ [16] [17]; it corresponds to twice the experimental uncertainty. It also corresponds to $\sin(\theta_{12}) = 0.2192$, while $V_{us} \equiv s_{12}c_{13}$ is measured to be $0.2247(19)$ [18] [17]; there, the discrepancy is $2/100$, only slightly above the $1.8/100$ relative width of the experimental interval.

The approximation which sets $\theta_{13} = 0 = \tilde\theta_{13}$ is accordingly reasonable, though it yields results slightly away from experimental bounds. We show in the next subsection that relaxing this approximation gives results in very good agreement with present experiments.

5.2 Going to $(\theta_{13} \neq 0, \tilde\theta_{13} \neq 0)$

Considering all angles to be Cabibbo-like with, in addition to (59)

$$\tilde{\theta}_{13} = \theta_{13} + \rho, \tag{65}$$

the l.h.s.'s of eqs. (51),(52),(53), (54), (55) and the sum (54 + 55) depart respectively from zero by

$$\eta c_{13}\left[s_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}c_{12}c_{23}s_{23}\right] - \rho c_{12}(c_{13}^2 - s_{13}^2); \tag{66a}$$

$$\eta c_{13}\left[-c_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}s_{12}c_{23}s_{23}\right] - \rho s_{12}(c_{13}^2 - s_{13}^2); \tag{66b}$$

$$-\epsilon(c_{12}^2 - s_{12}^2) + \eta\left[s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2) - 2c_{23}s_{23}c_{12}s_{12}(1 + s_{13}^2)\right] + 2\rho c_{13}s_{13}c_{12}s_{12}; \tag{66c}$$

$$4\epsilon c_{12}s_{12} + \eta\left[-4s_{13}s_{12}c_{12}(c_{23}^2 - s_{23}^2) - 2c_{23}s_{23}(c_{12}^2 - s_{12}^2)(1 + s_{13}^2)\right] + 2\rho c_{13}s_{13}(c_{12}^2 - s_{12}^2); \tag{66d}$$

$$-2\epsilon s_{12}c_{12} + \eta\left[2s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + 2c_{23}s_{23}\left((c_{12}^2 - s_{12}^2) + c_{13}^2(1 + s_{12}^2)\right)\right] + 2\rho c_{13}s_{13}(1 + s_{12}^2); \tag{66e}$$

$$2\epsilon s_{12}c_{12} + \eta\left[-2s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + 2c_{23}s_{23}\left(c_{13}^2(1 + c_{12}^2) - (c_{12}^2 - s_{12}^2)\right)\right] + 2\rho c_{13}s_{13}(1 + c_{12}^2). \tag{66f}$$

We have added (66f), which is not an independent relation, but the sum of (66d) and (66e); it expresses the violation in the universality of diagonal $\{11\}$ and $\{33\}$ currents.

5.2.1 A guiding calculation

Before doing the calculation in full generality, and to make a clearer difference with the neutrino case, we first do it in the limit where one neglects terms which are quadratic in the small quantities θ_{13} and ρ. By providing simple intermediate formulæ, it enables in particular to suitably choose the signs which occur in equating the moduli of two quantities. Eqs.(66) become

$$\eta\left[s_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}c_{12}c_{23}s_{23}\right] - \rho c_{12}; \tag{67a}$$

$$\eta\left[-c_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}s_{12}c_{23}s_{23}\right] - \rho s_{12}; \tag{67b}$$

$$-\epsilon(c_{12}^2 - s_{12}^2) + \eta\left[s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2) - 2c_{23}s_{23}c_{12}s_{12}\right]; \tag{67c}$$

$$4\epsilon c_{12}s_{12} - 2\eta\left[2s_{13}s_{12}c_{12}(c_{23}^2 - s_{23}^2) + c_{23}s_{23}(c_{12}^2 - s_{12}^2)\right]; \tag{67d}$$

$$-2\epsilon s_{12}c_{12} + 2\eta\left[s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + c_{23}s_{23}(1 + c_{12}^2)\right]; \tag{67e}$$

$$2\epsilon s_{12}c_{12} + 2\eta\left[-s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + c_{23}s_{23}(1 + s_{12}^2)\right]. \tag{67f}$$

The principle of the method is the same as before. From the relation (67c) = (-)(67d) [23] , which expresses that the absence of non-diagonal $\{12\}$ current is violated with the same strength as the universality of $\{11\}$ and $\{22\}$ currents, one gets ϵ/η as a function of $\theta_{12}, \theta_{23}, \theta_{13}$ [24]. This expression is plugged in the relation (67b) = (-)(67e)[25], which expresses the same condition for the $(2,3)$ channel; from this, one extracts ρ/η as a function of $\theta_{12}, \theta_{23}, \theta_{13}$ [26]. The expressions that have been obtained for ϵ/η and ρ/η are then inserted into the third relation, $|\,(67a)\,| = |\,(67f)\,|$, which now corresponds to the $(1,3)$ channel. This last step yields a relation $F_0(\theta_{12}, \theta_{23}, \theta_{13}) = 1$ between the three angles $\theta_{12}, \theta_{23}, \theta_{13}$.

[23] The (-) signs ensures that $\tan(2\theta_{12}) \approx (+)1/2$.

[24]

$$\frac{\epsilon}{\eta} = s_{13}(c_{23}^2 - s_{23}^2) + 2s_{23}c_{23}\frac{s_{12}c_{12} + c_{12}^2 - s_{12}^2}{4c_{12}s_{12} - (c_{12}^2 - s_{12}^2)}; \tag{68}$$

ϵ/η has a pole at $\tan(2\theta_{12}) = 1/2$, the suggested value of the Cabibbo angle for two generations.

[25] There, again, the (-) sign has to be chosen so as to recover approximately (62).

[26]

$$\frac{\rho}{\eta} = 2c_{23}s_{23}\left[s_{13} - c_{12}\left(2\frac{(c_{12}s_{12} + c_{12}^2 - s_{12}^2)}{4s_{12}c_{12} - (c_{12}^2 - s_{12}^2)} - \frac{1 + c_{12}^2}{c_{12}s_{12}} + \frac{1}{s_{12}}\frac{c_{23}^2 - s_{23}^2}{2s_{23}c_{23}}\right)\right]. \tag{69}$$

ρ/η has a pole at $\tan(2\theta_{12}) = 1/2$ and, for $\theta_{13} = 0$, it vanishes, as expected, when θ_{12} and θ_{23} satisfy the relation (62), which has been deduced for $\tilde{\theta}_{13} (\equiv \theta_{13} + \rho) = 0 = \theta_{13}$.

It turns out that $\frac{\partial F_0(\theta_{12},\theta_{23},\theta_{13})}{\partial\theta_{13}} = 0$, such that, in this case, a condition between θ_{12} and θ_{23} alone eventually fulfills the three relations under concern

$$1 = \left|\frac{\text{viol}([11]=[22])}{\text{viol}([12]=0=[21])}\right| = \left|\frac{\text{viol}([22]=[33])}{\text{viol}([23]=0=[32])}\right| = \left|\frac{\text{viol}([11]=[33])}{\text{viol}([13]=0=[31])}\right| \Leftrightarrow \tilde{F}_0(\theta_{12},\theta_{23}) = 1. \tag{70}$$

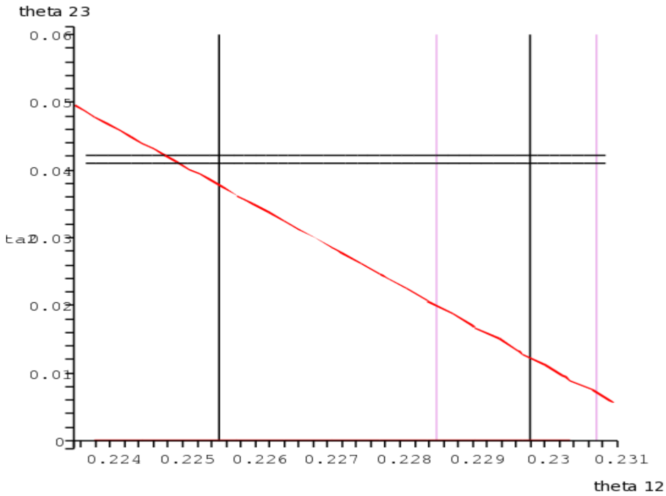

Fig. 4: θ_{23} for quarks as a function of θ_{12}; neglecting terms quadratic in θ_{13}

θ_{23} is plotted on Fig. 4 as a function of θ_{12}, together with the experimental intervals for θ_{23} (black horizontal lines) and θ_{12} (the intervals for θ_{12} come respectively from V_{ud} (eq. (63), black (dark) vertical lines) and V_{us} (eq. (64)), purple (light) vertical lines).

The precision obtained is much better than in Fig. 3 since, in particular, for θ_{23} within its experimental range, the discrepancy between the value that we get for θ_{12} and its lower experimental limit coming from V_{us} is smaller than the two experimental intervals, and even smaller than their intersection.

5.2.2 The general solution

The principle for solving the general equations (66) is the same as above. One first uses the relation (66c) = (-) (66d) to determine ρ/ϵ in terms of η/ϵ. The result is plugged in the relation (66b) = (-) (66e), which fixes η/ϵ, and thus ρ/ϵ as functions of $(\theta_{12},\theta_{23},\theta_{13})$. These expressions for η/ϵ and ρ/ϵ are finally plugged in the relation | (66a) | = | (66f) | , which provides a condition $F(\theta_{12},\theta_{23},\theta_{13}) = 1$. When it is fulfilled, the universality of each pair of diagonal neutral currents of mass eigenstates and the absence of the corresponding non-diagonal currents are violated with the same strength, in the three channels $(1,2)$, $(2,3)$ and $(1,3)$.

The results are displayed in Fig. 5; θ_{23} is plotted as a function of θ_{12} for $\theta_{13} = 0, 0.004$ and 0.01. Like in Figs. 3 and 4, the experimental bounds on θ_{12} are depicted by vertical black (dark) lines for the ones coming from V_{ud} and purple (light) for the ones coming from V_{us}; the experimental interval for θ_{23} corresponds to the black horizontal lines. The present experimental interval for θ_{13} is [17]

$$V_{ub} = \sin(\theta_{13}) \approx \theta_{13} \in [4\,10^{-3}, 4.6\,10^{-3}]. \tag{71}$$

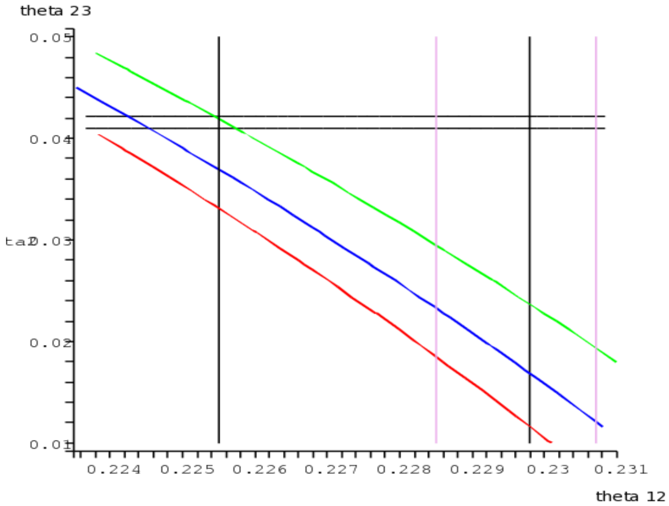

Fig. 5: θ_{23} for quarks as a function of θ_{12}, general case. $\theta_{13} = 0$ (red, bottom), 0.004 (blue, middle) and 0.01 (green, top)

We conclude that:

$*$ The discrepancy between our results and experiments is smaller than the experimental uncertainty;

$*$ a slightly larger value of θ_{13} and/or slightly smaller values of θ_{23} and/or θ_{12} still increase the agreement between our results and experimental measurements;

$*$ the determination of θ_{12} from V_{us} seems preferred to that from V_{ud}.

Another confirmation of the relevance of our criterion is given in the next section concerning neutrino mixing angles.

6 Beyond unitarity. A neutrino-like pattern; quark-lepton complementarity

In the "quark case", we dealt with three "Cabibbo-like" angles. The configuration that we investigate here is the one in which θ_{23} is, as observed experimentally [17], (close to) maximal, and θ_{12} and θ_{13} are Cabibbo-like (see subsection 4.3). The two cases only differ accordingly from the "classical solutions" of the unitarization equations away from which one makes small variations. The criterion to fix the mixing angles stays otherwise the same.

6.1 The case $\theta_{13} = 0 = \tilde\theta_{13}$

We explore the vicinity of this solution, slightly departing from the corresponding unitary mixing matrix, by considering that $\tilde\theta_{12}$ now slightly differs from θ_{12}, and $\tilde\theta_{23}$ from its maximal value

$$\begin{aligned} \tilde\theta_{12} &= \theta_{12} + \epsilon, \\ \theta_{23} = \pi/4 \quad , \quad \tilde\theta_{23} &= \theta_{23} + \eta. \end{aligned} \tag{72}$$

The l.h.s.'s of eqs. (51) (52) (53) (54) and (55) no longer vanish, and become respectively

$$-\frac{1}{2}\eta^2(s_{12}+\epsilon c_{12}), \tag{73a}$$

$$\frac{1}{2}\eta^2(c_{12}-\epsilon s_{12}), \tag{73b}$$

$$(*)\ -\eta s_{12}c_{12}+\epsilon(s_{12}^2-c_{12}^2)(1+\eta), \tag{73c}$$

$$(*)\ -\eta(c_{12}^2-s_{12}^2)+4\epsilon s_{12}c_{12}(1+\eta), \tag{73d}$$

$$\eta(1+c_{12}^2)-2\epsilon s_{12}c_{12}(1+\eta), \tag{73e}$$

showing by which amount the five conditions under scrutiny are now violated. Some care has to be taken concerning the accurateness of equations (73). Indeed, we imposed a value of θ_{13} which is probably not the physical one (even if close to). It is then reasonable to consider that channel $(1,2)$ is the less sensitive to this approximation and that, accordingly, of the five equations above, (73c) and (73d), marked with an "$*$", are the most accurate [27] .

The question: is there a special value of $\theta_{12}=\tilde\theta_{12}$ Cabibbo-like for which small deviations (ϵ,η) from unitarity entail equal strength violations of
$*$ the absence of $\{12\},\{21\}$ non-diagonal neutral currents;
$*$ the universality of $\{11\}$ and $\{22\}$ neutral currents ?

gets then a simple answer

$$s_{12}c_{12}=c_{12}^2-s_{12}^2 \Rightarrow \tan(2\theta_{12})=2. \tag{74}$$

We did not take into account the terms proportional to ϵ because we assumed that the mass splittings between the first and second generations (from which the lack of unitarity originates) are much smaller that the ones between the second and the third generation .

In the case of two generations, only ϵ appears, and one immediately recovers from (73c) and (73d) the condition fixing $\tan(2\theta_c)=1/2$ for the Cabibbo angle.

Accordingly, the same type of requirement that led to a value of the Cabibbo angle for two generations very close to the observed value leads, for three generations, to a value of the first mixing angle satisfying the quark-lepton complementarity relation (3) [6].

The values of θ_{12} and θ_{23} determined through this procedure are very close to the observed neutrino mixing angles [17] [19].

Though we only considered the two equations that are *a priori* the least sensitive to our choice of a vanishing third mixing angle (which is not yet confirmed experimentally), it is instructive to investigate the sensitivity of our solution to a small non-vanishing value of θ_{13}. This is done in Appendix E in which, for this purpose, we made the simplification $\tilde\theta_{13}\approx\theta_{13}$. It turns out that the terms proportional to s_{13} in the two equations $[12]=0=[21]$ and $|\,[11]\,|=|\,[22]\,|$ are also proportional to $(c_{23}^2-s_{23}^2)$, such that our solution with θ_{23} maximal is very stable with respect to a variation of θ_{13} around zero. This may of course not be the case for the other three equations, which are expected to be more sensitive to the value of θ_{13}.

6.2 Solutions for the angle θ_{13}

We now consider, like we did for quarks, the general case $\theta_{13}\neq 0\neq\tilde\theta_{13}(\rho\neq 0)$, $\tilde\theta_{12}\neq\theta_{12}(\epsilon\neq 0)$, $\tilde\theta_{23}\neq\theta_{23}(\eta\neq 0)$, while assigning to θ_{12} and θ_{23} their values obtained in subsection 6.1.

We investigate the eight different relations between θ_{12}, θ_{23} and θ_{13} which originate from the $2\times2\times2$ possible sign combinations in the conditions (70) (the r.h.s. is now replaced by a condition $F(\theta_{12},\theta_{23},\theta_{13})=1$ involving the three mixing angles), where each modulus can be alternatively replaced by "$+$" or "$-$".

[27] The limitation of this approximation also appears in the fact that (73b), of second order in η, is not compatible with (73e), which is of first order.

Among the solutions found for θ_{13}, only two (up to a sign) satisfy the very loose experimental bound

$$\sin^2(\theta_{13}) \leq 0.1. \tag{75}$$

They correspond respectively to the sign combinations $(+/-/-)$, $(+/+/+)$, $(-/+/+)$ and $(-/-/-)$

$$\begin{aligned} \theta_{13} = \pm 0.2717 \quad &, \quad \sin^2(\theta_{13}) = 0.072, \\ \theta_{13} = \pm 5.7\, 10^{-3} \quad &, \quad \sin^2(\theta_{13}) = 3.3\, 10^{-5}. \end{aligned} \tag{76}$$

The most recent experimental bounds can be found in [19]. They read

$$\sin^2(\theta_{13}) \leq 0.05, \tag{77}$$

which only leaves the smallest solution in (76) [28]. Future experiments will confirm, or infirm, for neutrinos, the properties that we have shown to be satisfied with an impressive accuracy by quark mixing angles.

7 Flavor transformations

Up to now, the observed "pattern" of flavor mixing has been disconnected from flavor symmetries. It has instead been connected with a precise scheme of departure from unitarity of the matrix controlling gauge neutral currents in bare flavor space. This contrasts with most approaches which, first, focus on mass matrices rather than gauge currents, secondly try to induce precise forms of the latter from horizontal symmetries [8]. The goal of this section is to (start to) span a bridge between the two. We shall investigate unitary flavor transformations, while restricting, for the sake of simplicity, to the case of two flavors, in which symmetry patterns are more conspicuous (the presence of a third generation has been seen, for example, to only lightly affect the Cabibbo angle).

The most natural unitary flavor group to be investigated is then $U(2)_f$, or $U(1)_f \times SU(2)_f$. For degenerate systems, this is a symmetry group of the Lagrangian. As soon as the degeneracy is lifted, it is no longer so, though an arbitrary unitary transformation on fermions should not change "physics" *i.e.* the physical masses and mixing angles. This last property means that unitary flavor transformations have to be considered from two points of view: on one side, we will check that physical mixing angles do not change when fermions are transformed, in particular that the process of renormalization by the counterterms of Shabalin goes also unaltered in the transformation, and, on the other side, we will investigate which changes they induce on the (different parts of) the Lagrangian, and how their breaking can eventually be associated with the pattern of neutral currents that seemingly controls mixing angles observed in nature.

For non-degenerate masses, the explicit form for the matrix $(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1}$ controlling neutral currents in the bare flavor basis, that has been obtained in section 2.1) (see 23)) provides an "orientation" of the relevant $SU(2)_f$ with respect to the trivial one (the generators of which are the Pauli matrices), which depends on the mixing angle θ: there arises the generator $\mathcal{T}_z(\theta)$. A trivial invariance of the effective Lagrangian of gauge neutral currents by transformation $e^{i(\alpha+\beta_z \mathcal{T}_z(\theta))}$ follows, which is broken for charged currents (unless the (d,s) and (u,c) sectors undergo identical transformations).

We then study possible connections between mass matrices and gauge neutral currents. We start by the simple case of a constant (symmetric) mass matrix. A link with neutral currents rapidly appears because, apart from trivial terms proportional to the unit matrix, the mass matrix is deduced from $(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1}$ by a translation $\theta \to \theta - \pi/4$ of the mixing angle. The departure of the mass matrix from (a term proportional to) unity is then represented by the $\mathcal{T}_x$ generator of the rotated $SU(2)_f(\theta)$ mentioned above. The commutator $\mathcal{T}_y = [\mathcal{T}_z(\theta), \mathcal{T}_x(\theta)]$ is the standard Pauli matrix, independent of the mixing angle,

[28] These values substantially differ from the ones in [20], which mainly focuses on special textures for the product of the quark and neutrino mixing matrices [21].

In a short paragraph, we single out a special invariance of both (non-trivial parts of) neutral currents and mass terms by the orthogonal, hermitian but non unitary transformations $e^{\alpha\mathcal{T}_y}$ (it is not a symmetry of the whole Lagrangian).

Then, we study general 2×2 unitary transformations on fermions. We demonstrate, through various steps, that these transformations go across Shabalin's renormalization and finally leave unchanged the renormalized mixing angles. Flavor rotations, equivalent to $e^{i\alpha\mathcal{T}_y}$ transformations appear of special relevance. They are shown to continuously rotate neutral currents into mass terms and to preserve their group structure. As far as charged currents are concerned, their group structure only stays unchanged when the same rotation is performed in the (u, c) ans (d, s) sectors. It then occurs that mass and flavor eigenstates can only be aligned in one of the two sectors. The mixing angle of the non-aligned sector becomes then identical to the Cabibbo angle occurring in charged currents. In this framework, as commented upon more at length in subsection 8.3, flavor rotations appear as a very mildly broken flavor subgroup of the electroweak Standard Model.

Last, we generalize this result to the renormalized mass matrix (fermionic self-energy). Its dependence on q^2 leads, like in the general argumentation of QFT used in [2], to the presence of an orthonormal basis of eigenstates for each value of q^2, containing one at most among the physical mass eigenstates. Accordingly, one reaches the same conclusion concerning the non-unitarity *a priori* of mixing matrices. The $U(1)_{em}$ Ward identity that connects the photon-fermion-antifermion vertex function at zero external momentum to the derivative of the inverse propagator requires that the two sides of the identity be invariant by the same transformation $e^{i(\alpha+\beta_z\mathcal{T}_z(\theta))}$ mentioned above. This yields a constraint that we propose to adopt because it guarantees that (q^2 dependent) neutral currents and the fermionic self energy keep the same structure as that encountered in the case of a constant mass matrix; they are in particular, again, continuously transformed into each other by flavor rotations. The resulting expressions are in particular, unlike textures, stable by these transformations.

7.1 A first type of horizontal symmetries

We exhibit below specific flavor transformations transformations that leave parts of the gauge electroweak Lagrangian invariant. We deal with the case of two generations, which makes an easy link with [3], and consider for example the (d, s) channel. The corresponding neutral currents in the basis of bare flavor eigenstates are controlled by the product $(\mathcal{C}_d^{-1})^\dagger\mathcal{C}_d^{-1}$.

When $\mathcal{C}_d$ departs from unitarity, we parametrize it like in (22) in which the role of A_d is now played by ϵ_d, such that (23) becomes

$$(\mathcal{C}_d^{-1})^\dagger(\theta_{dL})\mathcal{C}_d^{-1}(\theta_{dL}) = 1 + 2\epsilon_d\,\mathcal{T}_z(\theta_{dL}), \tag{78}$$

where the expression for $\mathcal{T}_z$ has been given in (16). Whatever be θ_{dL}, the unitary transformation

$$\Omega_z(\alpha_d, \beta_d, \theta_{dL}) = e^{i(\alpha_d+\beta_d\mathcal{T}_z(\theta_{dL}))} \tag{79}$$

with arbitrary α_d and β_d, acting on $\begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}$, satisfies

$$\Omega_z^\dagger(\alpha_d, \beta_d, \theta_{dL})\ \left[(\mathcal{C}_d^{-1})^\dagger(\theta_{dL})\mathcal{C}_d^{-1}(\theta_{dL})\right]\ \Omega_z(\alpha_d, \beta_d, \theta_{dL}) = (\mathcal{C}_d^{-1})^\dagger(\theta_{dL})\mathcal{C}_d^{-1}(\theta_{dL}), \tag{80}$$

and, thus, leaves invariant Lagrangian for gauge neutral currents

$$\begin{pmatrix} \bar d^0_{fL} & \bar s^0_{fL} \end{pmatrix} W^3_\mu\gamma^\mu_L \left[(\mathcal{C}_d^{-1})^\dagger(\theta_{dL})\mathcal{C}_d^{-1}(\theta_{dL})\right] \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}. \tag{81}$$

It is accordingly a horizontal group of invariance of the gauge Lagrangian for neutral currents in the space of bare flavor states [29].

As can be seen on (39), such transformations $\begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix} \rightarrow e^{i(\alpha_d+\beta_d \mathcal{T}_z(\theta_d))} \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}$, $\begin{pmatrix} u^0_{fL} \\ c^0_{fL} \end{pmatrix} \rightarrow e^{i(\alpha_u+\beta_u \mathcal{T}_z(\theta_{uL}))} \begin{pmatrix} u^0_{fL} \\ c^0_{fL} \end{pmatrix}$, acting independently in the (u,c) and (d,s) sectors (with different parameters), do not leave the gauge charged currents invariant.

A special invariance of the non-trivial parts of both neutral gauge currents and mass terms by a non-unitary transformation will also be exhibited in subsection 7.2.2.

7.1.1 The example of the Cabibbo angle $\tan 2\theta_c = \frac{1}{2}$

The value of the Cabibbo angle $\tan 2\theta_c = \frac{1}{2}$ [3] corresponds to $\sin 2\theta_c = \frac{1}{\sqrt{5}}$ and, so, $\mathcal{T}_z(\theta_c) = \dfrac{1}{2\sqrt{5}} \begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix} = \dfrac{1}{2} t_c, t_c^2 = 1$. In this case, the horizontal group of invariance of the corresponding neutral currents $(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1}$ in bare flavor space is, (α_d, β_d) being arbitrary parameters

$$\Omega_z(\alpha_d,\beta_d,\theta_c) = e^{i(\alpha_d+\beta_d t_c)} = e^{i\alpha_d}(\cos\beta_d + i t_c \sin\beta_d) = e^{i\alpha_d} \begin{pmatrix} \cos\beta_d + \frac{i}{\sqrt{5}}\sin\beta_d & -\frac{2i}{\sqrt{5}}\sin\beta_d \\ -\frac{2i}{\sqrt{5}}\sin\beta_d & \cos\beta_d - \frac{i}{\sqrt{5}}\sin\beta_d \end{pmatrix}. \tag{85}$$

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[29] Things become clearer in the proper basis of Ω_z, which is also the one of $\mathcal{T}_z(\theta_{dL})$. Its eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{1+\sin 2\theta_{dL}} \\ \sqrt{1-\sin 2\theta_{dL}} \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1-\sin 2\theta_{dL}} \\ \sqrt{1+\sin 2\theta_{dL}} \end{pmatrix} \tag{82}$$

and the diagonalized $\mathcal{T}_z(\theta_{dL})$ is

$$D_{T_z} = \frac{1}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = T^3. \tag{83}$$

Accordingly, in the proper (θ_{dL} dependent) basis, the horizontal group of invariance is

$$\Omega(\alpha_d,\beta_d) = e^{i(\alpha_d+\beta_d T^3)}, \tag{84}$$

that is, up to an arbitrary phase, a $U(1)$ transformation by the T^3 subgroup of the horizontal $SU(2)_f$ symmetry associated to the triplet of neutral currents in the (d,s) channel. In the proper basis, the neutral currents become controlled by $1+\epsilon_d D_{T_z} = 1+\epsilon_d T^3$, close to unity like in the mass basis and in the flavor basis.

The proper basis $\begin{pmatrix} d_p \\ s_p \end{pmatrix}$ of $(\mathcal{C}_d^{-1})^\dagger \mathcal{C}_d^{-1}, \mathcal{T}_z(\theta_{dL})$, and Ω_z can be easily expressed in terms of the renormalized mass basis (19) $\begin{pmatrix} d_p \\ s_p \end{pmatrix} = P_d^\dagger \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix} = P_d^\dagger \mathcal{C}_d \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}$, where P_d is the unitary matrix the columns of which are the eigenvectors (82): $P_d = P_d^\dagger = \begin{pmatrix} -\cos\zeta_d & \sin\zeta_d \\ \sin\zeta_d & \cos\zeta_d \end{pmatrix}$ with $\tan\zeta_d = \frac{\sqrt{1-\sin 2\theta_{dL}}}{\sqrt{1+\sin 2\theta_{dL}}} \Rightarrow \tan 2\zeta_d = \frac{1}{\tan 2\theta_{dL}} \Rightarrow \zeta_d = -\theta_{dL} + \frac{\pi}{4} + k\frac{\pi}{2}$. $\mathcal{C}_d$ being given by (13) and (22), $P_d^\dagger \mathcal{C}_d$ writes $\begin{pmatrix} -\cos(\theta_{dL}+\zeta_d) & \sin(\theta_{dL}+\zeta_d) \\ \sin(\theta_{dL}+\zeta_d) & \cos(\theta_{dL}+\zeta_d) \end{pmatrix} \equiv \begin{pmatrix} -\cos(\frac{\pi}{4}+k\frac{\pi}{2}) & \sin(\frac{\pi}{4}+k\frac{\pi}{2}) \\ \sin(\frac{\pi}{4}+k\frac{\pi}{2}) & \cos(\frac{\pi}{4}+k\frac{\pi}{2}) \end{pmatrix}$ up to corrections in ϵ_d.

Note that it is of the form $\begin{pmatrix} \alpha & -2(\alpha-\beta) \\ -2(\alpha-\beta) & \beta \end{pmatrix}$ like $(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1}$, belonging to the same group of matrices (see subsection 7.2.1).

7.2 Gauge currents versus mass matrices

In this work, the determination of mixing angles has been disconnected from the knowledge and / or assumptions concerning mass matrices, *e.g.* textures. In addition to the fact, already mentioned, that a single constant mass matrix cannot account for the properties of coupled systems in QFT [5][10], the limitations of putting the emphasis on mass matrices have already often been stressed. Textures are unstable by unitary transformations on fermions and cannot represent genuine physical properties of the system under consideration. In [22] it was explicitly shown how one can obtain, for example, bi-maximal mixing matrices without Dirac mass matrices playing any role. On these grounds, this last work casts serious doubts on the relevance of the Quark-Lepton Complementarity relation, which does not rely on "invariant" relations and quantities. That the Golden ratio value for $\tan\theta_c$ can be recovered from special textures (see for example [11]) can thus only be considered as a special case of some more general properties.

It is noticeable that the way we obtained these two properties stays independent of any assumption concerning mass matrices, since the remarkable properties at work concern gauge currents.

The problem that comes to mind is clearly whether a bridge can be spanned between gauge currents and some class of mass matrices. We just make a few remarks below; in a first step we shall consider a (abusively) single constant mass matrix; then, we will consider renormalized, q^2 dependent, mass matrices.

7.2.1 The case of a constant mass matrix

We have demonstrated in subsection 2.1 that non-unitary mixing matrices arise in the diagonalization of renormalized kinetic terms; this does not depend on the form of the classical mass matrix M_0. We consider, in a first step, the simple case of a binary system endowed with a real symmetric mass matrix

$$M_0 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \tag{86}$$

Calling m_1 and m_2 its eigenvalues, one can re-parametrize

$$\begin{aligned} M_0 &= m + \Delta m\, \mathcal{T}_x(\theta),\ \mathcal{T}_x(\theta) = \frac{1}{2}\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \\ m &= \frac{m_1+m_2}{2},\ \Delta m = m_1 - m_2, \end{aligned} \tag{87}$$

where θ is the (classical) mixing angle arising from the diagonalization of M_0. It satisfies

$$\tan 2\theta = \frac{2c}{a-b}. \tag{88}$$

That a given mixing angle can be related to infinitely many different mass patterns clearly appears since, for example, shifting M by $\kappa \times$ the unit matrix does not change the mixing angle, does not change Δm either and shifts each individual eigenvalue by κ. In particular, a value of κ much larger than $m_{1,2}$ leads to a quasi-degenerate binary system, the mixing angle of which stays nevertheless the same since $\tan 2\theta = 2c/\sqrt{(\Delta m)^2 - 4c^2}$ is unchanged. Also, two mass matrices proportional to each other have the same mixing angle though their eigenvalues have the same proportionality factor (mass ratios keep the same in this case) [30]. Trying to explain a given mixing pattern by a specific mass matrix is thus illusory because it cannot tackle the problem in its generality.

Shifting M_0 by a constant is a particular one among the transformations that leave the r.h.s. of (88) unchanged, *i.e.* the ones such that $\frac{2c}{a-b} = \text{cst}$. The set $\{\Theta(u)\}$ of such matrices [31]

$$\Theta(u) = \begin{pmatrix} a & \frac{u}{2}(a-b) \\ \frac{u}{2}(a-b) & b \end{pmatrix} = \frac{a+b}{2} + \frac{a-b}{2}\begin{pmatrix} 1 & u \\ u & -1 \end{pmatrix} \tag{89}$$

form, for any given u, a real abelian group, spanned by the two matrices 1 and $\frac{1}{2}\frac{1}{\sqrt{1+u^2}}\begin{pmatrix} 1 & u \\ u & -1 \end{pmatrix}$.

Interesting connections can be obtained as follows. Comparing (78) and (87), one gets:

$$\frac{(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1} - 1}{2\epsilon}(\theta) = \frac{M_0 - m}{\Delta m}(\theta - \pi/4). \tag{90}$$

It then appears natural to consider the three $SU(2)_f(\theta)$ generators (anticommuting matrices with eigenvalues $\pm 1/2$)

$$\mathcal{T}_x(\theta) = \frac{1}{2}\frac{1}{\sqrt{1+u^2}}\begin{pmatrix} 1 & u \\ u & -1 \end{pmatrix},\ \mathcal{T}_y = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix},\ \mathcal{T}_z(\theta) = \frac{1}{2}\frac{1}{\sqrt{1+u^2}}\begin{pmatrix} u & -1 \\ -1 & -u \end{pmatrix}, \tag{91}$$

such that, parametrizing [32]

$$\cos 2\theta = \frac{1}{\sqrt{1+u^2}},\ \sin 2\theta = \frac{u}{\sqrt{1+u^2}}, \tag{92}$$

one has, like in (87) and (78)

$$\begin{aligned} M_0 &= m + \Delta m\, \mathcal{T}_x(\theta), \\ (\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1} &= 1 + 2\epsilon\, \mathcal{T}_z(\theta). \end{aligned} \tag{93}$$

The $\vec{\mathcal{T}}(\theta)$'s are related to the standard $SU(2)$ generators $\vec{T}$ defined in (41) by

$$\begin{pmatrix} \mathcal{T}_x(\theta) \\ \mathcal{T}_z(\theta) \end{pmatrix} = R(u)\begin{pmatrix} T_x \\ T_z \end{pmatrix},\ R(u) = \frac{1}{\sqrt{1+u^2}}\begin{pmatrix} u & 1 \\ -1 & u \end{pmatrix};\ R^T(u)R(u) = 1. \tag{94}$$

"Mass terms" and neutral currents are transformed into one another by the action of (see (14) for the definition of the rotation $\mathcal{R}$)

$$e^{i\gamma \mathcal{T}_y} = \cos\frac{\gamma}{2} + 2i\mathcal{T}_y \sin\frac{\gamma}{2} = \mathcal{R}(\frac{\gamma}{2}). \tag{95}$$

Indeed,

$$\begin{pmatrix} e^{-i\gamma\mathcal{T}_y}\mathcal{T}_x(\theta)e^{i\gamma\mathcal{T}_y} \\ e^{-i\gamma\mathcal{T}_y}\mathcal{T}_z(\theta)e^{i\gamma\mathcal{T}_y} \end{pmatrix} = \mathcal{R}(-\gamma)\begin{pmatrix} \mathcal{T}_x(\theta) \\ \mathcal{T}_z(\theta) \end{pmatrix}, \tag{96}$$

[30] Any homographic transformation on a mass matrix M: $M \rightarrow \frac{\alpha M+\beta}{\delta M+\gamma}$ preserves the eigenvectors of M and thus the mixing angles.

[31] This set is of interest to us because, as we recalled in section 5, the Cabibbo angle empirically corresponds to $u \equiv \tan 2\theta_c = 1/2$, and, as we showed in sections 5 and 6, the same structure underlies, for three generations, quark and leptonic mixing angles. The empirical criterion equating the violation of universality and that of the absence of FCNC's corresponds to the same structure, in which the difference of diagonal elements of a symmetric 2×2 matrix is identical, up to a sign, to its off-diagonal one.

[32] For $u \equiv \tan 2\theta$ to be continuous, we have to restrict, for example, θ to the interval $]-\pi/4, +\pi/4[$.

which we rewrite [33]

$$\begin{pmatrix} \hat{\mathcal{T}}_x(\theta) \\ \hat{\mathcal{T}}_z(\theta) \end{pmatrix} = \mathcal{R}(-\gamma) \begin{pmatrix} \mathcal{T}_x(\theta) \\ \mathcal{T}_z(\theta) \end{pmatrix}, \quad \hat{\mathcal{T}}_{x,z}(\theta) = e^{-i\gamma\mathcal{T}_y}\mathcal{T}_{x,z}(\theta)e^{i\gamma\mathcal{T}_y}. \tag{98}$$

Comparing (95) and (98) shows that $e^{i\gamma T_y}$, which shifts θ by $\gamma/2$, rotates fermions by $\gamma/2$, but rotates the $\mathcal{T}_x(\theta)$ and $\mathcal{T}_z(\theta)$ generators by $(-\gamma)$. In particular, when rotating the fermions by $\pi/4$, *i.e.* taking $\gamma = \pi/2$, $\mathcal{T}_x(\theta) \to -\mathcal{T}_z(\theta),\ \mathcal{T}_z(\theta) \to \mathcal{T}_x(\theta)$.
Combining with (94), one finds

$$\begin{aligned} \begin{pmatrix} \hat{\mathcal{T}}_x(\theta) \\ \hat{\mathcal{T}}_z(\theta) \end{pmatrix} &= \begin{pmatrix} \sin(\gamma+2\theta) & \cos(\gamma+2\theta) \\ -\cos(\gamma+2\theta) & \sin(\gamma+2\theta) \end{pmatrix} \begin{pmatrix} T_x \\ T_z \end{pmatrix} \\ &= \mathcal{R}\left(-(\gamma+2\theta-\frac{\pi}{2})\right) \begin{pmatrix} T_x \\ T_z \end{pmatrix} \\ &= \begin{pmatrix} e^{-i(2\theta+\gamma-\frac{\pi}{2})T_y}\ T_x\ e^{i(2\theta+\gamma-\frac{\pi}{2})T_y} \\ e^{-i(2\theta+\gamma-\frac{\pi}{2})T_y}\ T_z\ e^{i(2\theta+\gamma-\frac{\pi}{2})T_y} \end{pmatrix}. \end{aligned} \tag{99}$$

The rotation matrix occurring is exactly of the same type as $R(u)$ occurring in (94), with its argument shifted from 2θ to $2\theta+\gamma$. (94) rewrites in particular

$$\begin{pmatrix} \mathcal{T}_x(\theta) \\ \mathcal{T}_y \\ \mathcal{T}_z(\theta) \end{pmatrix} = e^{-2i(\theta-\frac{\pi}{4})T_y} \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} e^{2i(\theta-\frac{\pi}{4})T_y}. \tag{100}$$

(91) shows that one recovers the standard $SU(2)$ generators T_x, T_y, T_z at the limit $u \to +\infty$ ($\theta \to \pi/4$); when $u \to -\infty$ ($\theta \to -\pi/4$), $\mathcal{T}_{x,z}(\theta) \to -T_{x,z}$; at the limit $u \to 0$ ($\theta = 0$), $\mathcal{T}_x(\theta) \to T_z, \mathcal{T}_z(\theta) \to -T_x$. By the transformation (isomorphic to Z_2) $u \to -1/u$, $\mathcal{T}_x(\theta) \to \mathcal{T}_z(\theta),\ \mathcal{T}_z(\theta) \to -\mathcal{T}_x(\theta)$. It corresponds to the transformation $\tan 2\theta \to -1/\tan 2\theta$, which is an outer automorphism of the $SU(2) \times U(1)$ (or $U(2)$) algebra under scrutiny. One can also speak of an infinite set of $SU(2)_f$, depending of the continuous parameter u. This set is divided by the transformation $u \to -1/u$ into two subsets, respectively with generators $\{\mathcal{T}_x(\theta), \mathcal{T}_y, \mathcal{T}_z(\theta)\}$ and $\{\mathcal{T}_z(\theta), \mathcal{T}_y, -\mathcal{T}_x(\theta)\}$. They intersect along the $U(1)$ group with generator $\mathcal{T}_y$, which is independent of θ.

7.2.2 A special invariance

In subsection 7.1, we encountered the unitary transformations Ω_z which leave invariant the Lagrangian of neutral currents. Likewise, we can define transformations $\Omega_x = e^{i(\alpha+\beta\mathcal{T}_x(\theta))}$, which, due to (87), leave mass terms invariant. None is a symmetry of both terms: neutral currents are not invariant by Ω_x, nor are mass terms by Ω_z.

[33] In terms of neutral currents and mass terms, one has

$$\begin{aligned} e^{-i\gamma\mathcal{T}_y} M_0 e^{i\gamma\mathcal{T}_y} &= m + \Delta m(\mathcal{T}_x(\theta)\cos\gamma - \mathcal{T}_z(\theta)\sin\gamma) \approx (m + \Delta m\mathcal{T}_x(\theta)) - \gamma\Delta m\mathcal{T}_z(\theta) \\ &= M_0 - \gamma\frac{\Delta m}{2\epsilon}\left((\mathcal{C}^{-1})^\dagger\mathcal{C}^{-1} - 1\right), \\ e^{-i\gamma\mathcal{T}_y}(\mathcal{C}^{-1})^\dagger\mathcal{C}^{-1}e^{i\gamma\mathcal{T}_y} &= 1 + 2\epsilon(\mathcal{T}_x(\theta)\sin\gamma + \mathcal{T}_z(\theta)\cos\gamma) \approx (1 + 2\epsilon\mathcal{T}_z(\theta)) + 2\epsilon\gamma\mathcal{T}_x(\theta) \\ &= (\mathcal{C}^{-1})^\dagger\mathcal{C}^{-1} + \gamma\frac{2\epsilon}{\Delta m}(M_0 - m). \end{aligned} \tag{97}$$

There exist a special invariance satisfied by the non-trivial parts of the mass matrix and of neutral currents, which results from the anticommutation of $\mathcal{T}_y$ with $\mathcal{T}_x$ and $\mathcal{T}_z$. Both $\dfrac{M_0 - m}{\Delta m}$ and $\dfrac{(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1} - 1}{2\epsilon}$ satisfy

$$O^\dagger \frac{M_0 - m}{\Delta m} O = \frac{M_0 - m}{\Delta m},\ O^\dagger \frac{(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1} - 1}{2\epsilon} O = \frac{(\mathcal{C}^{-1})^\dagger \mathcal{C}^{-1} - 1}{2\epsilon}, \tag{101}$$

where O is the orthogonal matrix depending on an arbitrary real parameter α

$$O = \left(\begin{array}{cc} \cosh\alpha & -i\sinh\alpha \\ i\sinh\alpha & \cosh\alpha \end{array}\right) \equiv e^{i\alpha\left(\begin{array}{cc} & -1 \\ 1 & \end{array}\right)} = e^{2\alpha\mathcal{T}_y},\ OO^T = 1,\ O^\dagger O \neq 1. \tag{102}$$

This transformation is non-unitary, such that the trivial parts of the matrices for mass and neutral currents (the ones proportional to the unit matrix) are not invariant.

It is also noticeable that the corresponding parts of the Lagrangian are *not* invariant by the unitary $U(1)$ rotation $\tilde{O}$ obtained by going to imaginary α. At the opposite, the trivial parts of the corresponding mass terms and gauge neutral currents, which are not invariant by O, are invariant by $\tilde{O}$.

7.2.3 Unitary transformations on fermions

Δm (mass splitting) and 2ϵ (lack of unitarity of the mixing matrix) cannot be but tightly connected; they are in particular expected to vanish simultaneously. When both vanish, the mixing angle is undetermined: mass terms, proportional to the unit matrix, are trivially invariant by the $SU(2)_f(\theta) \times U(1)_f$ flavor symmetry; so are the terms corresponding to neutral currents in the gauge Lagrangian.

As soon as the degeneracy is lifted, this symmetry is broken: mass terms and neutral currents are no longer invariant. However, it is the common belief that "physics" should not depend on arbitrary unitary flavor transformations on the fermion fields. So, on one side, we will check this point and, on the other side, we will study how different parts of the Lagrangian transform, putting a special emphasis on flavor rotations.

Flavor rotations

According to (95), they are strictly equivalent (up to a phase) to transformations $\Omega_y = e^{i(\alpha + \beta\mathcal{T}_y(\theta))}$. We consider

$$\left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right) \rightarrow \mathcal{R}(\varphi) \left(\begin{array}{c} d^0_{fL} \\ s^0_{fL} \end{array}\right), \tag{103}$$

which is equivalent to a transformation $e^{2i\varphi\mathcal{T}_y}$.

Concerning mass terms and neutral currents (in the original (bare) flavor basis), they respectively transform according to (see (78) and (87))

$$\mathcal{R}^\dagger(\varphi)\mathcal{T}_x(\theta)\mathcal{R}(\varphi) = \mathcal{T}_x(\theta + \varphi), \quad \mathcal{R}^\dagger(\varphi)\mathcal{T}_z(\theta)\mathcal{R}(\varphi) = \mathcal{T}_z(\theta + \varphi), \tag{104}$$

which consistently shifts the angle $\theta \rightarrow \theta + \varphi$. Such transformations, in particular, rotate continuously mass terms into neutral currents (see also (98)).

To ascertain that they "do not change physics" (given that they are obviously not symmetries of the Lagrangian), we must check that physical mixing angles are not changed by such transformations, in particular the Cabibbo angle occurring in charged currents of renormalized mass states. We accordingly consider (103) acting on (d^0_{fL}, s^0_{fL}), together with

$$\left(\begin{array}{c} u^0_{fL} \\ c^0_{fL} \end{array}\right) \rightarrow \mathcal{R}(\vartheta) \left(\begin{array}{c} u^0_{fL} \\ c^0_{fL} \end{array}\right). \tag{105}$$

By the unitary $\mathcal{R}(\vartheta,\varphi)$, the (u,c) classical mass matrix is left-multiplied by $\mathcal{R}^\dagger(\vartheta)$ and the (d,s) one by $\mathcal{R}^\dagger(\varphi)$. In the diagonalization process by a bi-unitary transformation, the unitary matrices $\mathcal{C}_{d0}$ and $\mathcal{C}_{u0}$ (see subsection 2.1) have simply to be changed into $\hat{\mathcal{C}}_{u0} = \mathcal{R}^\dagger(\vartheta)\mathcal{C}_{u0}$ and $\hat{\mathcal{C}}_{d0} = \mathcal{R}^\dagger(\varphi)\mathcal{C}_{d0}$ (in this simple case of rotations, the classical angles linking the original flavor states to the new mass states have become $\hat\theta_{uL} = \theta_{uL}+\vartheta$, $\hat\theta_{dL} = \theta_{dL}+\varphi$ [34]). So doing, the bare masses stay the same. The new classical mass eigenstates are $\begin{pmatrix} \hat u^0_{mL} \\ \hat c^0_{mL}\end{pmatrix} = \mathcal{C}^\dagger_{u0}\mathcal{R}(\vartheta)\begin{pmatrix} u^0_{fL} \\ c^0_{fL}\end{pmatrix}$, $\begin{pmatrix} \hat d^0_{mL} \\ \hat s^0_{mL}\end{pmatrix} = \mathcal{C}^\dagger_{d0}\mathcal{R}(\varphi)\begin{pmatrix} d^0_{fL} \\ s^0_{fL}\end{pmatrix}$; they are deduced from the original ones by the transformations $\mathcal{C}^\dagger_{u0}\mathcal{R}(\vartheta)\mathcal{C}_{u0} \equiv \mathcal{R}(\vartheta)$ and $\mathcal{C}^\dagger_{d0}\mathcal{R}(\varphi)\mathcal{C}_{d0} \equiv \mathcal{R}(\varphi)$. By the action of $\mathcal{R}(\vartheta)$ and $\mathcal{R}(\varphi)$, the classical charged currents Lagrangian becomes $\begin{pmatrix} \bar u^0_{fL} & \bar c^0_{fL}\end{pmatrix}\mathcal{R}^\dagger(\vartheta)\not{W}\mathcal{R}(\varphi)\begin{pmatrix} d^0_{fL} \\ s^0_{fL}\end{pmatrix}$, which writes in terms of the new classical mass eigenstates as (using the unitarity of $\mathcal{R}(\vartheta)$ and $\mathcal{R}(\varphi)$) $\begin{pmatrix} \overline{\hat u^0_{mL}} & \overline{\hat c^0_{mL}}\end{pmatrix}\mathcal{C}^\dagger_{u0}\not{W}\mathcal{C}_{d0}\begin{pmatrix} \hat d^0_{mL} \\ \hat s^0_{mL}\end{pmatrix}$. So, at the classical level, the mixing (Cabibbo) matrix occurring in charged currents is unchanged. This means that, in the equivalent of (30), involving the new classical mass eigenstates defined above, $\mathcal{C}_0$ is formally unchanged and so are the $SU(2)_L$ generators. This is precisely the ingredients which are used to calculate Shabalin's counterterms. So, in the new classical mass basis, the $A_{u,d}, B_{u,d}, D_{u,d}$'s are unchanged. This entails that the renormalized matrix $\mathcal{C}$ expressed by (32) is also unchanged. The last step is to go to the basis of the new renormalized mass eigenstates $\begin{pmatrix} \hat d_{mL} \\ \hat s_{mL}\end{pmatrix}$ (see (19)). Since the $A_{u,d}$ Shabalin's counterterms are unchanged, so are, formally, the matrices $\mathcal{V}_{u,d}$ (see (10,15)), which still depend on arbitrary angles φ_{Lu} and φ_{Ld} and parameters ρ_u and ρ_d. Since Shabalin's counterterms $B_{u,d}$ and $D_{u,d}$ are unchanged, so are the unitary matrices $V_{u,d}$ and $U_{u,d}$. Since $\mathcal{C}_{d0}$ and $\mathcal{C}_{u0}$ have been changed (see above), so have $\mathcal{C}_d$ and $\mathcal{C}_u$ (see (22)), in which θ_{dL} and θ_{uL} are now also respectively shifted by φ and ϑ. Let us keep as before $\varphi_{Lu}+\theta_{2Lu} = 0 = \varphi_{Ld}+\theta_{2Ld}$; $\mathcal{V}_d V_d$, which does not depend on θ_{dL} (see (20)), stays unchanged (and so does $\mathcal{V}_u V_u$). Since $\mathcal{C}$ has been seen to be unchanged, too, the Cabibbo matrix $\mathfrak{C}$, expressed by the first line of (34), is unchanged.

$\mathcal{C}_d$, which connects original flavor states to renormalized mass states, becomes $\hat{\mathcal{C}}_d \equiv \hat{\mathcal{C}}_{d0}\mathcal{V}_d V_d = \mathcal{R}^\dagger(\varphi)\mathcal{C}_d$ and one gets a similar expression for $\hat{\mathcal{C}}_u$.

We introduce, like before, renormalized flavor states (see (40)) and the renormalized mixing matrices $\hat{\mathfrak{C}}_u$ and $\hat{\mathfrak{C}}_d$ connecting the latter to renormalized mass states. Redoing the manipulations that led from (32) to (35), one finds that (35) stays unchanged. So do the three first terms of (42), as well as $(\mathfrak{C}_u)^{-1}$. The rotation angles φ and ϑ can be absorbed in the definition of the new renormalized flavor states which are, as expected, deduced from the initial ones (see (40)) by $\mathcal{R}(\varphi)$ and $\mathcal{R}(\vartheta)$. Finally, $\hat{\mathfrak{C}}_d = \mathfrak{C}_d$, $\hat{\mathfrak{C}}_u = \mathfrak{C}_u$: each renormalized mixing matrix stays unchanged and the relation $\mathfrak{C} \equiv \hat{\mathfrak{C}} = \hat{\mathfrak{C}}^\dagger_u\hat{\mathfrak{C}}_d$ still holds. The mixing angles are renormalized as before according to

$$\tilde\theta_{uL} = \theta_{uL} + \frac{\rho_u\epsilon_u}{2},\ \tilde\theta_{dL} = \theta_{dL} + \frac{\rho_d\epsilon_d}{2},\ \tilde\theta_c = \tilde\theta_{dL} - \tilde\theta_{uL}. \tag{106}$$

Let us also write what happens in there for charged currents (the transformations of mass terms and

[34] Since φ and ϑ are both free, by tuning the former to $-\theta_{dL}$ and the latter to $-\theta_{uL}$, one can tune both $\hat{\mathcal{C}}_{d0} = \mathcal{R}^\dagger(\varphi)\mathcal{C}_{d0}$ and $\hat{\mathcal{C}}_{u0} = \mathcal{R}^\dagger(\vartheta)\mathcal{C}_{u0}$ to the unit matrix: the mixing angles connecting, in both sectors, the new classical mass states to the initial flavor states, can thus be cast to zero (the mixing angles connecting the rotated bare flavor states to the new classical mass eigenstates are left unchanged by the rotation). However, if one considers charged currents in flavor space, we show after (107) that their group structure stays unaltered only if the two arbitrary flavor rotations become identical; this accordingly favors a common arbitrary flavor rotation in the two sectors. See also appendix F for the reverse statement that, by a flavor rotation, one can always align the new flavor states to the classical mass states.

neutral currents are given in (104)). It is convenient for this to use the second line of (38):

$$\begin{pmatrix} \bar u_{mL} & \bar c_{mL} \end{pmatrix} \mathfrak{C}\gamma^\mu \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix} = \begin{pmatrix} \bar u^0_{fL} & \bar c^0_{fL} \end{pmatrix} [1 + \epsilon_u \mathcal{T}_z(\theta_{uL}) + \epsilon_d \mathcal{T}_z(\theta_{dL})]\, \gamma^\mu \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}. \tag{107}$$

We recall (see subsection 2.1) that ϵ_d and ϵ_u are proportional to $\sin(\theta_{dL} - \theta_{uL})\cos(\theta_{dL} - \theta_{uL})$. By the transformations (103) and (105) the arguments of $\mathcal{T}_z(\theta_{uL})$ and $\mathcal{T}_z(\theta_{dL})$ in (107) are both shifted by $(\varphi + \vartheta)$: $2\theta_{uL} \to 2\theta_{uL} + \varphi + \vartheta$, and $2\theta_{dL} \to 2\theta_{dL} + \varphi + \vartheta$, such that their difference stays the same. The structure of (107) stays unchanged, but for the 1, which becomes $\mathcal{R}(\varphi - \vartheta)$ [35]. Because of this term, the group structure of charged currents is modified, since it no longer projects only on $\mathcal{T}_z$, unless $\varphi = \vartheta$; accordingly, if one wants to preserve it, the same flavor rotation should be performed in both sectors.

So, while independent $e^{i\alpha\mathcal{T}^u_y} \times e^{i\beta\mathcal{T}^d_y}$ flavor rotations do not change, in the new bases, the mixing angles (see also footnote 34), they modify the different parts of the Lagrangian in different ways. The tightly connected structure of neutral currents and mass terms stay unchanged and they are continuously rotated into one another. The modification of charged currents is more important unless the two rotations are identical. Accordingly, requesting that neutral and charged gauge currents exhibit the same flavor structure provides a constraint on the arbitrary flavor rotations that can be performed and thus a connection between sectors of different electric charge. This is one of the consequences of the fact that the angles of the two sectors get entangled by radiative corrections. It has also consequences for the alignment (up to small radiative corrections) of mass and flavor eigenstates in *one* of the two sectors (u, c) or (d, s) (see subsection 8.3).

Arbitrary unitary transformations

We now consider arbitrary 2×2 unitary transformations Ω^u and Ω^d on fermions.

Like for rotations, it is straightforward to show that $\mathcal{C}_0$, Shabalin's counterterms, $\mathcal{C}$, $\mathcal{V}_{u,d}$, $V_{u,d}$ and $U_{u,d}$ stay unchanged. So do $\mathcal{V}_d V_d$ and $\mathcal{V}_u V_u$ and, finally, the Cabibbo matrix $\mathfrak{C}$ between the new renormalized mass states. $\mathcal{C}_d$ becomes $\hat{\mathcal{C}}_d = \Omega^{d\dagger}\mathcal{C}_d$ and $\mathcal{C}_u$ becomes $\hat{\mathcal{C}}_u = \Omega^{u\dagger}\mathcal{C}_u$.

We parametrize, with the appropriate u or d index for α and $\vec\beta$

$$\Omega = e^{i(\alpha + \beta_x \mathcal{T}_x(\theta) + \beta_y \mathcal{T}_y + \beta_z \mathcal{T}_z(\theta))}. \tag{109}$$

Concerning mass terms and neutral currents, in the original flavor basis one gets:

$$\Omega^\dagger \mathcal{T}_x(\theta)\Omega \approx \mathcal{T}_x(\theta + \frac{\beta_y}{2}) + \beta_z \mathcal{T}_y, \quad \Omega^\dagger \mathcal{T}_z(\theta)\Omega \approx \mathcal{T}_z(\theta + \frac{\beta_y}{2}) - \beta_x \mathcal{T}_y, \tag{110}$$

while, for charged currents (107) becomes:

[35] One can check directly this statement by starting again from (34), which we have shown to be unchanged (though $\mathcal{C}_{u0}$ and $\mathcal{C}_{d0}$ have changed, $\mathcal{C}_0$ stays unchanged). We just have to make the transformation from the new classical mass eigenstates to the original flavor states. This is the role of the transformations $\hat{\mathcal{C}}_{d0}$ and $\hat{\mathcal{C}}_{u0}$ such that, in the original flavor basis the charged currents write (omitting the $W_\mu\gamma^\mu$)

$$\begin{aligned} &\begin{pmatrix} \bar u^0_{fL} & \bar c^0_{fL} \end{pmatrix} \hat{\mathcal{C}}_{u0} \frac{1}{2} \left[\begin{pmatrix} 1 & -A_u \\ -A_u & 1 \end{pmatrix} \mathcal{C}_0 + \mathcal{C}_0 \begin{pmatrix} 1 & -A_d \\ -A_d & 1 \end{pmatrix} \right] \hat{\mathcal{C}}^\dagger_{d0} \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix} \\ = &\begin{pmatrix} \bar u^0_{fL} & \bar c^0_{fL} \end{pmatrix} \mathcal{R}^\dagger(\vartheta) \left\{ \frac{1}{2}\, \mathcal{C}_{u0} \left[\begin{pmatrix} 1 & -A_u \\ -A_u & 1 \end{pmatrix} \mathcal{C}_0 + \mathcal{C}_0 \begin{pmatrix} 1 & -A_d \\ -A_d & 1 \end{pmatrix} \right] \mathcal{C}^\dagger_{d0} \right\} \mathcal{R}(\varphi) \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}, \end{aligned} \tag{108}$$

which yields the same conclusion as operating with $\mathcal{R}(\vartheta)$ and $\mathcal{R}(\varphi)$ directly on (107).

$$
\begin{aligned}
&\Omega^{u\dagger}\left[1+\epsilon_u \mathcal{T}_z(\theta_{uL})+\epsilon_d \mathcal{T}_z(\theta_{dL})\right]\Omega^d \\
\approx\ &\Omega^{u\dagger}\Omega^d \quad + \quad \epsilon_u\Bigg[\mathcal{T}_z\left(\theta_{uL}+\frac{\beta_y^u+\beta_y^d}{4}\right)+i(\alpha_d-\alpha_u)\mathcal{T}_z(\theta_{uL})-\frac{1}{2}\beta_x^u\,\mathcal{T}_y-\frac{i}{4}\beta_z^u \\
&\qquad\qquad\qquad\qquad +\frac{i}{4}\beta_x^d\,F\big((\theta_{uL}-\theta_{dL})\big)+\frac{i}{4}\beta_z^d\,G\big((\theta_{uL}-\theta_{dL})\big)\Bigg] \\
&\qquad\quad + \quad \epsilon_d\Bigg[\mathcal{T}_z\left(\theta_{dL}+\frac{\beta_y^u+\beta_y^d}{4}\right)+i(\alpha_d-\alpha_u)\mathcal{T}_z(\theta_{dL})-\frac{1}{2}\beta_x^d\,\mathcal{T}_y+\frac{i}{4}\beta_z^d \\
&\qquad\qquad\qquad\qquad +\frac{i}{4}\beta_x^u\,F\big((\theta_{uL}-\theta_{dL})\big)-\frac{i}{4}\beta_z^u\,G\big((\theta_{uL}-\theta_{dL})\big)\Bigg], \\
&\text{with } F(\tau)=\begin{pmatrix}\sin 2\tau & \cos 2\tau\\ -\cos 2\tau & \sin 2\tau\end{pmatrix},\quad G(\tau)=\begin{pmatrix}\cos 2\tau & \sin 2\tau\\ -\sin 2\tau & \cos 2\tau\end{pmatrix}. \qquad (111)
\end{aligned}
$$

So, mass terms, neutral currents and charged currents are all in general deeply modified, which corresponds to a strong breaking of the $SU(2)_f \times U(1)_f$ flavor symmetry.

7.2.4 Self energy, electromagnetic current and Ward identity

Departure from the inappropriate Wigner-Weisskopf approximation [5][10] can also be done by working with an effective renormalized q^2-dependent mass matrix (self-energy) $M(q^2)$.

The eigenvalues of $M(q^2)$ are now q^2-dependent, and are determined by the equation $\det[M(q^2)-\lambda(q^2)]=0$ [36]. Let them be $\lambda_1(q^2)\ldots\lambda_n(q^2)$. The physical masses satisfy the n self-consistent equations $q^2=\lambda_{1\ldots n}(q^2)$, such that $m_1^2=\lambda_1(m_1^2)\ldots m_n^2=\lambda_n(m_n^2)$. At each m_i^2, $M(m_i^2)$ has n eigenvectors, but only one corresponds to the physical mass eigenstate; the others are "spurious" states [5]. Even if the renormalized mass matrix is hermitian at any given q^2, the physical mass eigenstates corresponding to different q^2 belong to as many different orthonormal sets of eigenstates and thus, in general, do not form an orthonormal set. The discussion proceeds like in the core of the paper, leading to similar conclusions.

We study below the role of the $U(1)_{em}$ Ward Identity connecting the inverse fermionic propagator $S^{-1}(q)$ to the photon-fermion-antifermion vertex $\Gamma_\mu(q,q)$ at vanishing incoming photon momentum. In each sector of (bare) flavor space, the vertex function is (due to the Gell-Mann-Nishijima relation between neutral $SU(2)_L$ and $U(1)_{em}$ generators in the standard model, and up to $\gamma^\mu\times$ the electric charge in the given sector) nothing more than $(\mathcal{C}^{-1})^\dagger(q^2)\mathcal{C}^{-1}(q^2)$ encountered before for neutral currents (see also footnote 13). Requesting that the two sides of the identity be invariant by the same flavor transformation (79) will induce constraints which do not suffer the major drawback of textures, their instability by such transformations.

In each channel, for example (d,s), the aforementioned Ward Identity writes

$$\Gamma_\mu(q,q)=\frac{\partial}{\partial q_\mu}S^{-1}(q). \qquad (112)$$

Accordingly, both sides of (112) should be invariant by the same group of symmetry.

We write the (d,s) propagator $S(q^2)$ (we suppose that it is symmetric, such that left and right eigenstates are obtained by the same rotation) as

$$S^{-1}(q^2)=\not{q}-M(q^2),\ M(q^2)=\begin{pmatrix}a(q^2) & c(q^2)\\ c(q^2) & b(q^2)\end{pmatrix}. \qquad (113)$$

[36] This is the simple case of a normal mass matrix, which can be diagonalized by a single (q^2-dependent) unitary matrix. When it is non-normal, the standard procedure uses a bi-unitary diagonalization.

Defining $\theta(q^2)$ such that $\tan 2\theta(q^2) = \dfrac{2c(q^2)}{a(q^2)-b(q^2)}$, $M(q^2)$ can then be rewritten

$$M(q^2) = \frac{a(q^2)+b(q^2)}{2} + \frac{a(q^2)-b(q^2)}{2\cos 2\theta(q^2)}\, \mathcal{T}_x(\theta(q^2)). \tag{114}$$

Differentiating both sides of (113) with respect to q_μ and using (114) yields

$$\begin{aligned}\frac{\partial}{\partial q_\mu} S^{-1}(q) \quad &= \quad \gamma^\mu + 2q_\mu \Bigg[\frac{\partial(a(q^2)+b(q^2))}{2\,\partial q^2} - \frac{a(q^2)-b(q^2)}{2\cos 2\theta(q^2)}\, \mathcal{T}_x(\theta(q^2))\frac{\partial\theta(q^2)}{\partial q^2} \\ &\quad + \left[\frac{\partial}{\partial q^2}\left(\frac{a(q^2)-b(q^2)}{2\cos 2\theta(q^2)}\right)\right]\ \mathcal{T}_z(\theta(q^2))\Bigg],\end{aligned} \tag{115}$$

in which only the first two terms, respectively proportional to the unit matrix and to $\mathcal{T}_z$, are invariant by the same transformation Ω_z (79) as $(\mathcal{C}^{-1})^\dagger\mathcal{C}^{-1}$ which controls both neutral and electromagnetic gauge currents; the last term, proportional to $\dfrac{\partial\Delta m(q^2)}{2\,\partial q^2}, m(q^2) = \dfrac{\lambda_+(q^2)+\lambda_-(q^2)}{2}$ (see footnote 37), is not. The invariance can be recovered if we constrain this derivative to vanish, that is the self-energy to satisfy the condition

$$a(q^2) - b(q^2) = 2\mu\cos 2\theta(q^2),\ \mu = \text{cst}, \tag{116}$$

(of course trivially satisfied for $a(q^2) = b(q^2)$, in which case $\theta(q^2) = \pi/4$) or, equivalently [37]

$$M(q^2) = a(q^2) - \mu\cos 2\theta(q^2) + \mu\ \mathcal{T}_x(\theta(q^2)). \tag{118}$$

Unlike textures, this form of the self-energy is stable by flavor rotations $\begin{pmatrix} d^0_{fL} \\ s^0_{fL}\end{pmatrix} \rightarrow \mathcal{R}(\varphi)\begin{pmatrix} d^0_{fL} \\ s^0_{fL}\end{pmatrix}$; $M(q^2)$ is transformed into $a(q^2) - \mu\cos 2\theta(q^2) + \mu\ \mathcal{T}_x(\theta(q^2)+\varphi)$, which shows that the mixing angle $\theta(q^2)$ has simply become, as expected, $\theta(q^2)+\varphi$ while the spectrum is unchanged. So is the form (78) for the vertex function Γ_μ.

Our conjecture is accordingly that any self-energy or vertex function should be of the form

$$\Xi(q^2) + \mu\begin{pmatrix} \cos 2\theta(q^2) & \pm\sin 2\theta(q^2) \\ \pm\sin 2\theta(q^2) & -\cos 2\theta(q^2)\end{pmatrix} \text{ or } \Sigma(q^2) + \mu\begin{pmatrix} \sin 2\theta(q^2) & \pm\cos 2\theta(q^2) \\ \pm\cos 2\theta(q^2) & -\sin 2\theta(q^2)\end{pmatrix}, \tag{119}$$

which make them stable by flavor rotations. They are in particular normal, and thus can always be diagonalized by a unique unitary transformation, which can be used to define both left and right eigenvectors.

Eq.(118) trivially rewrites

$$M(q^2) = a(q^2) + \mu\begin{pmatrix} 0 & \sin 2\theta(q^2) \\ \sin 2\theta(q^2) & -2\cos 2\theta(q^2)\end{pmatrix}, \tag{120}$$

reminiscent, up to $a(q^2)$ (which does not change $\theta(q^2)$) of the triangular matrix suggested in [11] for $\tan 2\theta(q^2) = -2$; however, while the expressions (119) are stable by flavor rotations, this particular texture is not. Indeed, rotating (118) and (120), one gets respectively

---

[37] The eigenvalues of $M(q^2)$ are $\lambda_+(q^2) = a(q^2) + 2\mu\sin^2\theta(q^2)$ and $\lambda_-(q^2) = a(q^2) - 2\mu\cos^2\theta(q^2)$ (thus $\mu = \frac{\lambda_+(q^2)-\lambda_-(q^2)}{2}$), such that the physical masses (poles of the propagator) satisfy

$$m_1 = a(m_1^2) + 2\mu\sin^2\theta(m_1^2),\ m_2 = a(m_2^2) - 2\mu\cos^2\theta(m_2^2). \tag{117}$$

The degenerate case $m_1 = m_2$ corresponds to $\mu = 0$. By (116), this is equivalent to $a(q^2) = b(q^2)$ and to $\theta = \frac{\pi}{4}$. For quasi-degenerate systems $m_1 \approx m_2 \approx m$, one has $\dfrac{m_1-m_2}{m_1+m_2} = \dfrac{\mu}{a(m^2)-\mu\cos 2\theta(m^2)} \approx \dfrac{\mu}{a(m^2)}$ and $\mu \approx \dfrac{m_1-m_2}{2}$.

$$R^\dagger(\varphi)\left[a(q^2) - \mu\cos 2\theta(q^2) + \mu\,\mathcal{T}_x(\theta(q^2))\right] R(\varphi) = a(q^2) - \mu\cos 2\theta(q^2) + \mu\,\mathcal{T}_x(\theta(q^2)+\varphi), \tag{121}$$

$$R^\dagger(\varphi)\left[a(q^2) + \mu\begin{pmatrix} 0 & \sin 2\theta(q^2) \\ \sin 2\theta(q^2) & -2\cos 2\theta(q^2)\end{pmatrix}\right] R(\varphi)$$
$$= a(q^2) + \mu\begin{pmatrix} -\sin 2\theta(q^2)\sin 2\varphi - 2\cos 2\theta \sin^2\varphi & \sin 2(\theta(q^2)+\varphi) \\ \sin 2(\theta(q^2)+\varphi) & \sin 2\theta(q^2)\sin 2\varphi - 2\cos 2\theta(q^2)\cos^2\varphi \end{pmatrix}. \tag{122}$$

By evaluating the ratio between twice the non-diagonal term and the difference of diagonal ones, one finds, on both (121) and (122), that, as expected, the mixing angle has become $\theta(q^2)+\varphi$. However, while the "structure" of (121) is manifestly preserved by the rotation, the 0 texture in (122) is not.

8 Conclusion, open issues and outlook

8.1 Summary

That mixing matrices connecting flavor to mass eigenstates of non-degenerate coupled fermion systems should not be considered *a priori* as unitary has been given in this work, in addition to general QFT arguments, a perturbative basis from the calculation of radiative corrections at 1-loop to fermionic self-energies and neutral currents. The counterterms of Shabalin, in particular kinetic counterterms (wave function renormalization), have been shown to play an important role, controlling the departure from 1 of the matrix of neutral currents in bare flavor space.

We have shown that, in the renormalized mass basis, which, unlike the bare one, is no longer orthonormal, the renormalized mixing (Cabibbo) matrix stays unitary and, as required by the closure of the $SU(2)_L$ gauge algebra, neutral currents are, like in the bare mass basis, controlled by the unit matrix.

The peculiar feature that is satisfied for two generations by the Cabibbo angle, that universality of neutral currents is violated with the same strength as the absence of FCNC's, has been shown to be compatible with all mixing angles of quarks and leptons for three generations, too. For neutrinos, we have shown that there exists only one solution for θ_{13} to the corresponding equations that rigorously falls within present experimental limits, and we have obtained, without any hypothesis (textures) concerning mass matrices, the property of "quark-lepton complementarity" between the Cabibbo angle and their θ_{12}.

Flavor symmetries, and their entanglement with $SU(2)_L$ gauge symmetry, have been shown to underlie the physics of mixing angles. In particular, for two generations, the ways gauge currents and fermionic mass terms (or self-energy) transform by flavor rotations bear common footprints left by a non-degenerate mass spectrum.

8.2 Comparison with previous works

At this stage, it can be useful to stress that, in this approach to the renormalization of mixing matrices, both kinetic and mass terms + counterterms have been simultaneously diagonalized. Having dealt with self-mass as well as wave function renormalization, the mixing matrices that we define connect bare mass states to renormalized mass states which do not anymore undergo non-local non-diagonal transitions.

This is not the case of previous approaches, in particular of [23], in which the sole diagonalization of mass terms + counterterms defines renormalized mass states; so there still exist among them non-diagonal kinetic-like transitions [38].

[38] In the renormalization scheme proposed in [23], it is furthermore impossible to cancel finite contributions to self-masses in all channels. As a results, in some of them, finite non-diagonal fermionic mass counterterms stay present, which, when inserted on external legs of $Wq_1\bar{q}_2$ vertex, can trigger right-handed currents at $\mathcal{O}(g^5)$ in the standard model.

We have shown that Shabalin's kinetic counterterms are the ones that drive the non-unitarity of mixing matrices. Would we have left them aside, like in [23], we would have reached the same conclusion as theirs, that renormalized individual mixing matrices are unitary.

The mechanism that keeps unitary the CKM or PMNS matrices occurring in charged current is thus different from the one advocated in [23]; it results from subtle cancellations between two individually non-unitarity mixing matrices and the fact that, because of $SU(2)_L$ gauge invariance which dictates the form of covariant derivatives, the customary expression $K = K_u^\dagger K_d$ for the CKM matrix in terms of individual mixing matrices for u- and d-type quarks is no longer valid.

Another important feature of our work is that the general QFT argument leading to non-unitary mixing matrices makes use of pole masses. These are the only ones which are gauge independent. This choice goes along with the existence of several q^2 scales. One can instead choose to consider the renormalized mass matrix (self-energy) at a given unique q^2, and to define the renormalized mass eigenstates through a bi-unitary diagonalization of this mass matrix. This leads to unitary mixing matrices. However, the renormalized masses that appear by this procedure (which are not the eigenvalues of the mass matrix) do not match the poles of the renormalized propagator (which correspond to different values of q^2). Because of this, they certainly cannot satisfy the criterion of gauge independence.

As for the fate of $SU(2)_L$ Ward Identities in a multiscale renormalization approach, in addition to the fact that $SU(2)_L$ gauge invariance is compatible with the existence of different mass scales, we refer to [24]: the regularization method might violate some invariance (gauge, Lorentz ...); one has then to introduce counterterms which violate it, too. After the regularization has been taken away, the S-matrix so obtained satisfies the requested invariance. There appears accordingly to be no fundamental obstacle (only technical difficulties) if the regularization procedure does not respect the Ward Identities corresponding to the invariance of the theory.

8.3 Physically relevant mixing angles

The results that have been exposed are valid for fermions of both electric charges. They concern the mixing angles which parametrize

$*$ for quarks, the mixing matrix K_u of u-type quarks as well as K_d of d-type quarks;

$*$ for leptons, the mixing matrix K_ν of neutrinos as well as that of charged leptons K_ℓ,

and we have shown that our approach accounts for the observed values of the mixing angles.

However, a question arises : the measured values of the mixing angles are commonly attached, not to a single mixing matrix, e.g. K_u or K_d, but to the product $K = K_u^\dagger K_d$ which occurs in charged currents when both quark types are mass eigenstates. Thus, in the standard approach, they are *a priori* related to an entanglement of the mixing angles of quarks (or leptons) of different charges. Then, if mixing angles in each sector are expected to satisfy the same criterion, their difference, which makes up, up to small approximation, the Cabibbo angle, would be expected to vanish.

The same issue arises in the leptonic sector. Let us consider for example the case of solar neutrinos: the flux of "electron neutrinos" detected on earth is (roughly) half the one predicted by solar model to be emitted from the sun. Would the flux predicted in solar models concern flavor neutrinos, and would also the detection process counts flavor neutrinos, the sole mixing matrix which controls their evolution and oscillations would be K_ν; it is indeed the only matrix involved in the projection of flavor states onto mass (propagating) states. The situation is different if the comparison is made between the (emitted and detected) fluxes of states ν_e, ν_μ, ν_τ defined in subsection 3.1; since their projections on the mass eigenstates now involve the product $K_\ell^\dagger K_\nu$, their oscillations are, like for quarks, controlled by an entanglement of the mixing angles of neutrinos and charged leptons. The nature of the neutrino eigenstates that are produced and detected is also sometimes questioned (see also for example [25]). An often proposed solution is that, for charged leptons, their flavor is defined to coincide with their mass [26], which amounts to setting $K_\ell = 1$.

This is indeed the solution that comes naturally to the mind since, as we stated in subsection 7.2.3 (flavor rotations) (see also appendix F): while arbitrary independent flavor rotations are a priori allowed in each sector of different charge, with the corollary statement that the only physically relevant mixing angles are the ones occurring in the Cabibbo matrix, these two rotations are constrained to be identical if one likes to preserve the group structure (breaking pattern) of both neutral and charged currents in bare flavor space. For $\vartheta = -\theta_{uL} = \varphi$, only the mixing angles in the (u,c) sector becomes vanishing (alignment of bare mass and flavor states in this sector). The structure (107) of charged gauge currents in bare flavor space becomes $\big(1+\epsilon_u(\theta_{uL}=0)\mathcal{T}_z(0)+\epsilon_d(\theta_{uL}=0)\mathcal{T}_z(\theta_{dL}-\theta_{uL})\big)$. Now, as discussed in subsection 2.1.1, the value of the parameter $\epsilon_u \equiv A_u$ depends on the renormalization scheme; for example its values in MS and $\overline{MS}$ differ by a constant proportional to $\gamma g^2 \sin\theta_c\cos\theta_c(m_s^2-m_d^2)/M_W^2$, γ being the Euler constant. A "physical" renormalization scheme [39] would correspond to the condition $\epsilon_u(\theta_{uL}=0)=0$. In this scheme, a classical unit mixing matrix (vanishing classical mixing angle) would not be modified by (1-loop) radiative corrections: mass and flavor eigenstates could keep aligned in the corresponding sector both at the classical level and at 1-loop [40]. Then, after aligning mass and flavor states in the (u,c) sector, that is, in practice, turning θ_u to zero by a flavor rotation, the formula (107) for charged current in bare flavor space becomes $\begin{pmatrix}\bar u_{mL} & \bar c_{mL}\end{pmatrix}\mathfrak{C}\gamma^\mu\begin{pmatrix} d_{mL}\\ s_{mL}\end{pmatrix} = \begin{pmatrix}\bar u^0_{fL} & \bar c^0_{fL}\end{pmatrix}\left[1+\epsilon_d(\theta_{uL}=0)\mathcal{T}_z(\theta_{dL}-\theta_{uL})\right]\gamma^\mu\begin{pmatrix} d^0_{fL}\\ s^0_{fL}\end{pmatrix}$, in which the only argument of the $\mathcal{T}_z$ generator is the Cabibbo angle. The criterion linking universality and FCNC's could accordingly be applied to charged gauge currents in the bare flavor basis, which controls the observed Cabibbo angle.

8.4 Shabalin's counterterms in the calculation of physical transitions

As far as physics is concerned, some remarks are due concerning decays like $K\to\pi\nu\bar\nu, \mu\to e\gamma, \mu\to e\nu\bar\nu$, for which 1-loop flavor changing neutral currents play an important role. One could indeed wonder what are the consequences on these transitions of the introduction of Shabalin's counterterms.

The first way to proceed is the usual one: no counterterm "à la Shabalin" is introduced and calculations are done in the bare mass basis, which is orthonormal as soon as the bare flavor basis is supposed to be so.

However, $d^0_m \leftrightarrow s^0_m$ transitions occur at 1-loop, which can be considered to jeopardize the standard CKM phenomenology [7]. To remedy this, the $A_{u,d}, B_{u,d}, E_{u,d}, D_{u,d}$ counterterms are added, and should then be included in any perturbative calculation. This second possibility may be cumbersome, due to their twofold nature (kinetic and mass) and the fact that they have both chiralities. Furthermore, for d and s off mass-shell (which occurs at 2-loops and more), their action can be no longer reduced to the cancellation of non-diagonal $d^0_m \leftrightarrow s^0_m$ transitions.
Note that the one-loop amplitude of, for example, $s^0_m \to d^0_m Z$ or $s^0_m\to d^0_m\gamma$ transition does not change when these counterterms are introduced since, on one side, they kill $s^0_m \to Z(\gamma)s^0_m \to Z(\gamma)d^0_m$ and $s^0_m\to d^0_m\to Z(\gamma)d^0_m$ transitions but, on the other side, the covariant derivative associated with the $\not p$ in Eq. (5) restores them (see Appendix B). So, the standard (without counterterms) 1-loop calculation of

[39] Its existence is only a conjecture. One could simply subtract from $A_u^{\overline{MS}}$ its value at $\theta_{uL}=0$, that is, another constant like when going from MS to $\overline{MS}$. However it is not clear that such a scheme respects the gauge Ward Identities, nor how to implement it in practice at the level of individual Feynman diagram. Subtracting from each one its value at $\theta_{uL}=0$ is the simplest choice as a "physical" renormalization prescription suitable to the alignment of flavor and mass states in the (u,c) sector. When applied to Fig. 1 or to its equivalent for $u^0_m \leftrightarrow c^0_m$ transitions (hence, in practice, to the functions $f_{u,d}$ (see footnote 14)), it also modifies the values of the $B_{u,d}, E_{u,d}, D_{u,d}$ counterterms and that of the combination (27), which keeps non-vanishing, because the subtracted constants have, in this case, a dependence on fermion masses and on p^2 more involved than the sole difference of $(mass)^2$ that factorizes the Euler constant γ in (24).

[40] The resulting mixing matrix, which is identical to the unit matrix, trivially satisfies the criterion under consideration, *i.e.* that the violation of universality (presently non-existing) is equal to that of the absence of FCNC's (also vanishing).

FCNC's stays valid when counterterms are introduced, and the latter do not accordingly play, there, any physical role.

The third possibility is to diagonalize the bare Lagrangian + Shabalin's counterterms and to perform calculations in the so-defined renormalized mass basis, in which it has the standard canonical form except that, as usual when counterterms are introduced, the parameters (masses, mixing angles) become the renormalized ones.

This form of the Lagrangian is extremely simple since all effects of Shabalin's counterterms have been re-absorbed in a renormalization of the masses and mixing angles, and a change of status (orthogonality or not) of the mass basis. However, again, non-diagonal $d_m \leftrightarrow s_m$ transitions can occur at 1-loop, between renormalized mass states. They are similar to the ones occurring in the bare Lagrangian without counterterms, except that their amplitudes are now expressed in terms of renormalized parameters. Two attitudes are then possible:

* either one applies standard Feynman rules to this Lagrangian without worrying about the orthonormality of the basis of reference, which leads back to the usual way of performing calculations; this is tantamount to considering that Shabalin's counterterms do not play any physical role. This can look a reasonable attitude since one does not know *a priori* whether a set of vectors is orthonormal or not, except on physical grounds;
* or, before starting any perturbative calculation, one first worries whether the reference basis is orthonormal or not. This is now tantamount to considering that physical predictions could depend on this property and that any sensible Lagrangian should be written, before any perturbative expansion is performed, in an orthonormal reference basis. These considerations go however beyond the scope of the present work.

8.5 Flavor rotations as a very softly broken symmetry of the Standard Model

Performing a rotation by an angle φ in the two sectors (u,c) and (d,s) (or (e,μ) and (ν_e,ν_μ)):
* shifts both arguments θ_{uL} and θ_{dL} in the $SU(2)_f$ generators $\mathcal{T}_z(\theta_{uL,dL})$ and $\mathcal{T}_x(\theta_{uL,dL})$ which occur respectively in neutral (and electromagnetic) currents and mass matrices by φ (see (104));
* yields equivalent shifts in charged currents (see (107));
* does not modify the physical Cabibbo angle (see "Flavor rotations" in subsection 7.2.3);
* leaves invariant the rest of the Standard Model Lagrangian and does not change the physical masses.

The rotation angle φ and the resulting modifications of the Lagrangian appear unphysical. This is why flavor rotations (identical in the two sectors) can be considered to be a symmetry of the Standard Model.

8.6 CP violation

In this work we have deliberately ignored CP violating mixing angles and all effects of CP violation. There are several reasons for this:
* they are *a priori* small and should not quantitatively alter the results that have been obtained for the other type of mixing angles;
* since the renormalized Cabibbo matrix is constrained by $SU(2)_L$ gauge invariance to stay unitary, we do not expect strong deviations from the customary results;
* the introduction of CP violating phases would considerably complicate the trigonometric equations to solve, which are already highly non-trivial.

There is however an interesting point: the most general non-unitary mixing matrix allows *a priori* CP violation even for two generations. But we consider this as another matter which deserves a separate investigation.

8.7 Open issues. Beyond the Standard Model

The present work raises several questions and challenges.

A first type of challenge concerns experimentally observable consequences of the issues raised in this work, specially the *a priori* non-unitarity of mixing matrices. Unlike the CP-violating parameters ϵ_L and ϵ_S of neutral kaons the difference of which we could estimate in [5], we are not yet able to exhibit and estimate observables which would be sensitive to this non-unitarity, or, equivalently, to the energy dependence of eigenstates induced by radiative corrections. This is all the more challenging as we have shown that the renormalized mixing matrix occurring in charged currents (Cabibbo, CKM, PMNS) keeps unitary as a consequence of $SU(2)_L$ gauge invariance. So, no deviation from unitarity can be expected in charged currents from this mechanism. The finite renormalization of mixing angles in charged currents by the simple function $\frac{\rho_d\epsilon_d-\rho_u\epsilon_u}{2}$ of Shabalin's counterterms is itself non-physical since the parameters $\rho_{u,d}$ are arbitrary. Non-unitary is thus expected to only be at work in neutral currents in bare flavor space, where only one fermionic sector gets involved. However, it is a much debated issue whether individual mixing angles, corresponding to a given sector, are observable, or whether the sole observable angles are the ones occurring in charged currents.

A connection should also be made with the non-unitary equivalence of mass and flavor Fock spaces investigated in [27]. We have shown that renormalized mass states are *a priori* connected to bare flavor space by non-unitary transformations, which preaches in favor of the propositions in [27]. However, we have also proved in subsection 2.2.3 that one can define renormalized flavor states which deduce from bare flavor states by a non-unitary transformation and which, now, connect to renormalized mass states by unitary transformations. The issue arises accordingly (see also subsection 8.4) of which basi(e)s can be considered to be orthonormal. Renormalized mass states and bare flavor states we have shown cannot be simultaneously orthonormal. Since, and this is the point of view of [27], physical mass states (that is renormalized mass states) are expected to have the standard anticommutation relations and to form a Fock space of orthonormal states, renormalized flavor states, which are unitarily connected to the latter, would then form, too, an orthonormal basis (such that bare flavor states should not be anymore considered to form an orthonormal set, nor bare mass states). Then the two spaces of renormalized flavor states and renormalized mass states would be two unitarily connected Fock spaces. This issue is currently under investigation.

It is to be mentioned that, often, mixing angles are not defined, like we did, through fundamental parameters of the Lagrangian, but as ratios of amplitudes among physical bound states (mesons). The connection between the two approaches is certainly to be investigated, but it is clear that it faces the tedious problem of bound states, in which any tentative calculation is doomed to uncertainties largely exceeding the effects that need to be tested.

The last type of challenge concerns the criterion that seemingly controls observed mixing angles: it connects in the simplest possible way the violation of unitarity to FCNC's in bare flavor space. We have no reason to believe that the Standard Model possesses, in itself, even including the refinements of QFT that we have implemented, the necessary ingredients to give birth to such a property. All it can tell is that, due to non-degeneracy, one expects both violation of unitarity and the presence of FCNC's for all gauge currents in bare flavor space. So, it seems reasonable to think that the realm of any possible connection between the two lies "beyond the Standard Model", and that only there can one hope to ultimately find a theoretical explanation to the observed pattern, and to the relation between the $\tan$ of the Cabibbo angle and the Golden ratio [3][11].

8.8 Conclusion and perspective

This work does not, obviously, belong to what is nowadays referred to as "Beyond the Standard Model", since it does not incorporate any "new physics" such as supersymmetry, "grand unified theories (GUT)" or extra-dimensions. However it does not strictly lie within the SM either, even if it is very close to. Of course, it shares with the latter its general framework (mathematical background and physical content), and also borrows from it the two physical conditions of universality for diagonal neutral currents and absence of FCNC's, which play a crucial role in the process. But, on the basis of the most general arguments of QFT, we make a decisive use of the essential non-unitarity of the mixing matrices, whereas only unitary matrices are present in the SM. This property may be considered, in the SM, as an ”accidental” characteristic of objects which are intrinsically non-unitary.

The mixing angles experimentally observed get constrained in the vicinity of this “standard” situation, a slight departure from which being due to mass splittings. Hence our approach can be considered to explore the ”Neighborhood of the Standard Model”, which is likely to exhibit low-energy manifestations of physics ”Beyond the Standard Model”.

While common approaches limit themselves to guessing symmetries for mass matrices (see for example [8] and references therein), we showed that relevant patterns reveal instead themselves in the violation of properties attached to gauge currents: in each given (i, j) flavor channel, two dimensional flavor rotation appears as a flavor subgroup softly broken by the presence of mass splittings, which continuously connects neutral currents and the fermionic self-energy.

When two generations are concerned, nature seems to exhibit a quantization of the $\tan$ of twice the mixing angles as multiples of $1/2$. This corresponds to the property that, in the original flavor basis, the effects of lifting the mass degeneracy are such that universality for neutral currents is violated with the same strength as the absence of FCNC's. The third generations appears as a small perturbation of this property. Whether this quantization really exists and whether it can be cast on a firm theoretical background, in particular through perturbative calculations, stays unfortunately an open question.

It is remarkable that the same type of symmetry underlies both the quark and leptonic sectors; they only differ through the $0th$ order solution to the “unitarization equations”, the twofold-ness of which was recently uncovered in [2]. In the neutrino case, the values that we obtain for the mixing angles (with the smallest one of θ_{13}) do not deviate by more than 10% from the tri-bimaximal pattern [28].

To conclude, this work demonstrates that flavor physics offers to our investigation very special and simple patterns which had been, up to now, unnoticed. Strong arguments in favor of them have been given in both the quark and leptonic sectors, and they will be further tested when the third mixing angle of neutrinos is accurately determined.

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Appendix

A Calculation of Shabalin's counterterms

We derive here the expressions (6) for Shabalin' s counterterms A_d, B_d, E_d, D_d.
Requesting that the sum of (4) and (5) vanishes for s_m^0 one mass-shell gives the first equation

$$f_d(p^2 = m_s^2)\bar{d}_m^0(1+\gamma^5)m_s\, s_m^0 = \\ A_d\, \bar{d}_m^0(1+\gamma^5)m_s\, s_m^0 + B_d\, \bar{d}_m^0(1-\gamma^5)s_m^0 + E_d\, \bar{d}_m^0(1-\gamma^5)m_s\, s_m^0 + D_d\, \bar{d}_m^0(1+\gamma^5)s_m^0. \tag{123}$$

Likewise, using $\bar{d}\,\overleftrightarrow{\partial_\mu} s \equiv \bar{d}(\partial_\mu s) - (\partial_\mu \bar{d})s$ and $i\gamma^\mu \partial_\mu \bar{s} = -m_s \bar{s}$, the equivalent request for d_m^0 on mass-shell yields

$$f_d(p^2 = m_d^2)\bar{d}_m^0 m_d(1-\gamma^5)s_m^0 = \\ A_d\, \bar{d}_m^0 m_d(1-\gamma^5)s_m^0 + B_d\, \bar{d}_m^0(1-\gamma^5)s_m^0 + E_d\, \bar{d}_m^0 m_d(1+\gamma^5)s_m^0 + D_d\, \bar{d}_m^0(1+\gamma^5)s_m^0. \tag{124}$$

(123) yields the two conditions, respectively for $(1+\gamma^5)$ and $(1-\gamma^5)$ terms:

$$\begin{aligned} m_s f_d(p^2 = m_s^2) &= m_s A_d + D_d, \\ B_d + m_s E_d &= 0. \end{aligned} \tag{125}$$

while (124 yields the two other conditions

$$\begin{aligned} m_d f_d(p^2 = m_d^2) &= m_d A_d + B_d, \\ D_d + m_d E_d &= 0. \end{aligned} \tag{126}$$

The solutions of the four equations in (125) and (126) are given by (6).

B The inclusion of Shabalin's counterterms does not modify $s \to d\gamma$ transition.

Making use of formula (4) for the 1-loop $s_m^0 \to d_m^0$ transition of Fig. 1, the left and center diagrams of Fig. 6 write respectively (we omit the ϵ^μ of the photon and use and abbreviated notation $f_d(m_d^2) = f_d(p^2 = m_d^2, m_u^2, m_c^2, m_W^2)$)

$$\bar{d}_m^0(p) f_d(m_s^2)\, \not{p}(1-\gamma^5)\, \frac{1}{\not{p} - m_s}\gamma^\mu s_m^0(p+q), \tag{127}$$

$$\bar{d}_m^0(p)\gamma_\mu\, \frac{1}{\not{p} - \not{q} - m_d} f_d(m_s^2)(\not{p} + \not{q})(1-\gamma^5)s_m^0(p+q). \tag{128}$$

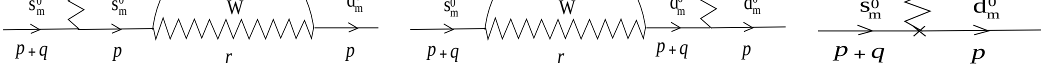

Fig. 6: diagrams contributing to $s_m^0 \to d_m^0\gamma$ sensitive to Shabalin's counterterms, which cancel the left and center amplitudes; the latter are re-created via the covariant derivatives inside A_d and E_d which yield the diagram on the right.

Using that the d quark is on mass-shell in Fig. 6 left and the s quark is on mass-shell in Fig. 6 center, straightforward manipulations transform (127) and (128) respectively into

$$f_d(m_d^2) \quad \left[\frac{m_d^2}{m_d^2-m_s^2}\bar{d}_m^0\gamma_\mu(1-\gamma^5)s_m^0 + \frac{m_d m_s}{m_d^2-m_s^2}\bar{d}_m^0\gamma_\mu(1+\gamma^5)s_m^0\right], \tag{129}$$

$$- \quad f_d(m_s^2) \quad \left[\frac{m_s^2}{m_d^2-m_s^2}\bar{d}_m^0\gamma_\mu(1-\gamma^5)s_m^0 + \frac{m_d m_s}{m_d^2-m_s^2}\bar{d}_m^0\gamma_\mu(1+\gamma^5)s_m^0\right], \tag{130}$$

the sum of which yields $A_d\, \bar{d}_m^0\, \gamma_\mu(1-\gamma^5)s_m^0 + E_d\, \bar{d}_m^0\, \gamma_\mu(1-\gamma^5)s_m^0$, where A_d and E_d are the Shabalin's counterterms given in (6).

So, while the two corresponding amplitudes are canceled by Shabalin's counterterms (since, in both diagrams, a $s_m^0 \to d_m^0$ transition occurs with either s or d on mass-shell), the photonic parts in the covariant derivatives which should be used inside A_d and E_d re-create the same transition amplitude (Fig. 6 right). $s_m^0 \to d_m^0\gamma$ is thus left unchanged by the introduction of these counterterms.

The same demonstration holds for $s_m^0 \to d_m^0 Z$ transitions.

C $\tilde{\theta}_{13} = 0 \Rightarrow \theta_{13} = 0$

Using the notations of section 3, we start with the following system of equations:

$$\frac{[11]+[22]}{2} = [33] \Leftrightarrow s_{13}^2 + s_{23}^2 + \tilde{c}_{23}^2 = 1; \tag{131a}$$

$$[11] = [22] \Leftrightarrow c_{13}^2\cos(2\theta_{12}) = (c_{23}^2+\tilde{s}_{23}^2)\cos(2\tilde{\theta}_{12}); \tag{131b}$$

$$[12] = 0 = [21] \Leftrightarrow c_{13}^2\sin(2\theta_{12}) = (c_{23}^2+\tilde{s}_{23}^2)\sin(2\tilde{\theta}_{12}); \tag{131c}$$

$$[13] = 0 = [31] \Leftrightarrow \tilde{s}_{12}\left(\sin(2\theta_{23}) - \sin(2\tilde{\theta}_{23})\right) = c_{12}\sin(2\theta_{13}); \tag{131d}$$

$$[23] = 0 = [32] \Leftrightarrow \tilde{c}_{12}\left(\sin(2\tilde{\theta}_{23}) - \sin(2\theta_{23})\right) = s_{12}\sin(2\theta_{13}). \tag{131e}$$

From equation (131a), we have $c_{23}^2 + \tilde{s}_{23}^2 \neq 0$, which entails $c_{13}^2 \neq 0$[41]. Let us study the consequence on the two equations (131b) and (131c).

• the two sides of (131b) vanish for $\cos(2\theta_{12}) = 0 = \cos(2\tilde{\theta}_{12})$, *i.e.* $\theta_{12} = \frac{\pi}{4}[\frac{\pi}{2}] = \tilde{\theta}_{12}$.
(131c) then gives $c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2$, which, associated with (131a), yields the following solution [42]: $\theta_{13} = 0[\pi]$ and $\tilde{\theta}_{23} = \pm\theta_{23}[\pi]$.

• the two sides of (131c) vanish for $\sin(2\theta_{12}) = 0 = \sin(2\tilde{\theta}_{12}) = 0$, i.e. $\theta_{12} = 0[\frac{\pi}{2}] = \tilde{\theta}_{12}$.
(131b) gives then $c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2$, hence, like previously, $\theta_{13} = 0[\pi]$ and $\tilde{\theta}_{23} = \pm\theta_{23}[\pi]$.

• in the other cases we can calculate the ratio (131b) / (131c), which gives $\tan(2\theta_{12}) = \tan(2\tilde{\theta}_{12})$, hence $\theta_{12} = \tilde{\theta}_{12}[\pi]$ or $\theta_{12} = \frac{\pi}{2} + \tilde{\theta}_{12}[\pi]$:

 $*$ $\theta_{12} = \frac{\pi}{2} + \tilde{\theta}_{12}[\pi]$ implies for (131b)(131c) $c_{13}^2 = -c_{23}^2 - \tilde{s}_{23}^2$, which, together with (131a) ($c_{13}^2 = s_{23}^2 + \tilde{c}_{23}^2$), gives a contradiction : $2 = 0$:

 $*$ $\theta_{12} = \tilde{\theta}_{12}(\neq 0)[\pi]$ implies, like previously, $c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2$, which gives, when combined with (131a): $\theta_{13} = 0[\pi]$ and $\tilde{\theta}_{23} = \pm\theta_{23}[\pi]$.

Hence, it appears that whatever the case, the solution gives rise to $\theta_{13} = 0[\pi]$.

Let us now look at (131d) and (131e). Since $\theta_{13} = 0$, the two r.h.s.'s vanish, and we obtain the twin equations $\tilde{s}_{12}(\sin(2\theta_{23}) - \sin(2\tilde{\theta}_{23})) = 0$ and $\tilde{c}_{12}(\sin(2\theta_{23}) - \sin(2\tilde{\theta}_{23})) = 0$, which, together, imply $\sin(2\theta_{23}) = \sin(2\tilde{\theta}_{23})$. It follows that, either $\theta_{23} = \tilde{\theta}_{23}[\pi]$ or $\theta_{23} = \frac{\pi}{2} - \tilde{\theta}_{23}[\pi]$;

[41] Indeed, let us suppose that c_{13} vanishes. Then $\cos(2\tilde{\theta}_{12})$ and $\sin(2\tilde{\theta}_{12})$ must vanish simultaneously, which is impossible.

[42] $$\left\{\begin{array}{l} c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2 \\ s_{13}^2 + s_{23}^2 + \tilde{c}_{23}^2 = 1 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} s_{23}^2 + \tilde{c}_{23}^2 = 1 \\ s_{13}^2 = 0 \end{array}\right.$$

$* \ \theta_{23} = \tilde\theta_{23}[\pi]$ matches the result of the previous discussion in the "+" case, whereas, in the "-" case, the matching leads to $\theta_{23} = \tilde\theta_{23} = 0$, which is to be absorbed as a particular case in the "+" configuration;

$* \ \theta_{23} = \frac{\pi}{2} - \tilde\theta_{23}[\pi]$ matches the result of the previous discussion in the "+" configuration, in which case it leads to $\theta_{23} = \tilde\theta_{23} = \frac{\pi}{4}[\frac{\pi}{2}]$, *i.e.* maximal mixing between the fermions of the second and third generations.

D $(\theta_{12}, \theta_{23})$ solutions of eqs. (51) (52) (53) (54) (55) for $\theta_{13} = 0 = \tilde\theta_{13}$

Excluding $\tilde\theta_{12} = 0$, (57a) and (57b) require $\sin(2\theta_{23}) = \sin(2\tilde\theta_{23}) \Rightarrow \tilde\theta_{23} = \theta_{23} + k\pi$ or $\tilde\theta_{23} = \pi/2 - \theta_{23} + k\pi$.

• for $\tilde\theta_{23} = \theta_{23} + k\pi$ Cabibbo-like,

(57c) requires $\sin(2\theta_{12}) = \sin(2\tilde\theta_{12}) \Rightarrow \tilde\theta_{12} = \theta_{12} + n\pi$ or $\tilde\theta_{12} = \pi/2 - \theta_{12} + n\pi$;

(57d) requires $\cos(2\theta_{12}) = \cos(2\tilde\theta_{12}) \Rightarrow \tilde\theta_{12} = \pm\theta_{12} + p\pi$;

(57e) requires $s_{12}^2 + \tilde c_{12}^2 - 1 = 0 \Rightarrow \tilde\theta_{12} = \pm\theta_{12} + r\pi$.

The solutions of these three equations are $\theta_{12} = \tilde\theta_{12} + m\pi$ Cabibbo-like or $\theta_{12} = \pi/4 + q\pi/2$ maximal ($\tilde\theta_{12} = \pm\theta_{12} + r\pi$ is then also maximal). They are associated with $\tilde\theta_{23} = \theta_{23} + k\pi$, condition heading this paragraph.

• for $\tilde\theta_{23} = \pi/2 - \theta_{23} + k\pi$,

(57c) requires $s_{12}c_{12} = 2c_{23}^2 \tilde s_{12}\tilde c_{12}$;

(57d) requires $c_{12}^2 - s_{12}^2 = 2c_{23}^2(\tilde c_{12}^2 - \tilde s_{12}^2)$;

(57e) requires $s_{12}^2 + 2c_{23}^2\tilde c_{12}^2 - 2s_{23}^2 = 0$.

Taking the ratio of the first two conditions yields $\tan(2\theta_{12}) = \tan(2\tilde\theta_{12}) = 2c_{23}^2 \Rightarrow \tilde\theta_{12} = \theta_{12} + k\pi/2 + n\pi$, which entails $2c_{23}^2 = 1 \Rightarrow \theta_{23} = \pm\pi/4 + p\pi/2$ maximal; by the condition $\tilde\theta_{23} = \pi/2 - \theta_{23} + k\pi$ heading this paragraph, $\tilde\theta_{23}$ is then maximal, to. The third condition becomes $s_{12}^2 + \tilde c_{12}^2 - 1 = 0$, which requires $\tilde\theta_{12} = \pm\theta_{12} + r\pi$. Then, the second condition is automatically satisfied, but the first requires that the "$+$" sign be chosen; so, $\tilde\theta_{12} = \theta_{12} + r\pi$ is Cabibbo-like.

• Summary: the solutions are:

$* \ \tilde\theta_{23} = \theta_{23} + k\pi$ Cabibbo-like, associated with either $\theta_{12} = \tilde\theta_{12} + m\pi$ Cabibbo-like or θ_{12} and $\tilde\theta_{12}$ maximal;

$* \ \tilde\theta_{12} = \theta_{12} + r\pi$ Cabibbo-like, associated with θ_{23} and $\tilde\theta_{23}$ maximal.

E Sensitivity of the neutrino solution to a small variation of θ_{13}

If one allows for a small $\theta_{13} \approx \tilde\theta_{13}$, (53) and (54) become respectively

$$-2\eta s_{12}c_{12}s_{23}c_{23} + \epsilon(s_{12}^2 - c_{12}^2) + \eta s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2) = 0 \tag{132}$$

and

$$-2\eta s_{23}c_{23}(c_{12}^2 - s_{12}^2) + 4\epsilon s_{12}c_{12} - 2\eta s_{13}(c_{23}^2 - s_{23}^2)(2s_{12}c_{12} + \epsilon(c_{12}^2 - s_{12}^2)) = 0. \tag{133}$$

For $\theta_{23}, \tilde\theta_{23}$ maximal, the dependence on θ_{13} drops out.

F Aligning classical flavor states and classical mass states

We show below that, at the classical level of mass matrices, one can always perform, in each sector, a flavor rotation such that the classical mass eigenstates and the rotated flavor states get aligned. Since the logic is slightly different from the one in paragraph 7.2.3 [43], we chose to explain things in detail here.

Let us now consider the *change of variables* in flavor space $\begin{pmatrix} d'_{fL} \\ s'_{fL} \end{pmatrix} = \mathcal{R}(\varphi) \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix}$. In terms of the primed fields, the mass terms in the Lagrangian rewrite $\begin{pmatrix} d'_{fL} \\ s'_{fL} \end{pmatrix}^\dagger \mathcal{R}(\varphi) M_0 \mathcal{S}^\dagger(\varphi) \begin{pmatrix} d'_{fR} \\ s'_{fR} \end{pmatrix}$, in which $\mathcal{S}(\varphi)$ is the equivalent of $\mathcal{R}(\varphi)$ for right-handed fields. Since M_0 was diagonalized according to $\mathcal{C}^\dagger_{d0} M_0 \mathcal{H}_{d0} = diag(m^0_d, m^0_s)$, $\mathcal{R}(\varphi) M_0 S^\dagger_\varphi$ is now diagonalized according to $\mathcal{C}^\dagger_{d0} R^\dagger_\varphi (\mathcal{R}(\varphi) M_0 \mathcal{S}^\dagger(\varphi)) \mathcal{S}(\varphi) \mathcal{H}_{d0} = diag(m^0_d, m^0_s)$. Accordingly, the new classical mass eigenstates are $\begin{pmatrix} d'_{mL} \\ s'_{mL} \end{pmatrix} = \mathcal{C}^\dagger_{d0} \mathcal{R}^\dagger(\varphi) \begin{pmatrix} d'_{fL} \\ s'_{fL} \end{pmatrix} = \mathcal{C}^\dagger_{d0} \begin{pmatrix} d^0_{fL} \\ s^0_{fL} \end{pmatrix} \equiv \begin{pmatrix} d^0_{mL} \\ s^0_{mL} \end{pmatrix}$. So, the classical mass eigenstates are unchanged, but are now deduced from the new classical flavor states by the product $\mathcal{C}^\dagger_{d0} \mathcal{R}^\dagger(\varphi)$. The angle φ can accordingly be tuned such that this product is the unit matrix. When it is so, the new classical flavor states are aligned with the bare mass states.

The same demonstration holds in the (u, c) sector. This shows that, at the classical level of mass matrices, mixing angles in each sector, when defined as the one connecting bare flavor states to original bare mass states have no physical meaning and can always be tuned to zero. So, the only physical mixing angles are the ones occurring in charged currents. Indeed, since mass states are unchanged, it is even more trivial than in subsection 7.2.3 to show that these angles stay unchanged by arbitrary flavor rotations.

We recall however that, as emphasized in footnote 34, a common flavor rotation of both sectors is required as soon as one wants to preserve the group structure of charged currents in bare flavor space.

[43] The change in flavor states was defined, there, by (103) and the transformed Lagrangian was expressed in terms of the original bare flavor fields. Classical mass eigenstates got changed such that the new ones are deduced from the starting ones by the rotation $\mathcal{R}(\varphi)$: $\begin{pmatrix} \hat{d}^0_m \\ \hat{s}^0_m \end{pmatrix} = \mathcal{R}(\varphi) \begin{pmatrix} d^0_m \\ s^0_m \end{pmatrix}$. The new classical mass eigenstates could then be aligned with the starting bare flavor states. In the present approach, it is the new flavor states which can get aligned with the bare mass eigenstates, the latter staying unchanged.

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